

EXPANSION OF GENERALIZED STIELTJES CONSTANTS IN TERMS OF DERIVATIVES OF HURWITZ ZETA-FUNCTIONS.

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ABSTRACT. Generalized Stieltjes constants $\gamma_n(a)$ are the coefficients in the Laurent series for the Hurwitz-zeta function $\zeta(s, a)$ at the pole $s = 1$. Many authors proved formulas for these constants. In this paper, using a recurrence between $(\zeta(s + j, a))_j$ and proved by the author, we prove a general result which contains some of these formulas as particular cases.

1. INTRODUCTION

For $a \in \mathbb{C}$, $\Re(a) > 0$, the Hurwitz zeta function is defined as

$$\zeta(\sigma, a) = \sum_{k=0}^{\infty} \frac{1}{(k+a)^\sigma}, \quad \Re(\sigma) > 1.$$

It can be analytically continued to $\sigma \in \mathbb{C} \setminus \{1\}$, with a pole at $\sigma = 1$. The generalized Stieltjes constants $\gamma_n(a)$ occur as coefficients in the Laurent series expansion of $\zeta(\sigma, a)$ at the pole $\sigma = 1$:

$$\zeta(\sigma, a) = \frac{1}{\sigma - 1} + \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \gamma_\ell(a) (\sigma - 1)^\ell. \quad (1)$$

For $a = 1$, $\zeta(\sigma, a)$ is the usual zeta-function $\zeta(\sigma)$.

There are numerous representations for them, for instance [3, 9],

$$\gamma_\ell(a) = \lim_{m \rightarrow \infty} \left\{ \sum_{k=0}^m \frac{\ln^\ell(k+a)}{k+a} - \frac{\ln^{\ell+1}(m+a)}{\ell+1} \right\}, \quad \ell = 0, 1, 2, \dots, a \neq 0, -1, -2, \dots$$

If $a = 1$, the generalized Stieltjes constant $\gamma_\ell(a)$ is the usual Stieltjes constant γ_ℓ .

The series for Hurwitz zeta function converges absolutely for $\Re(\sigma) = \alpha > 1$ and the convergence is uniform in the half plane $\alpha \geq \alpha_0 > 1$. So $\zeta(\sigma, a)$ is analytic in the half-plane $\Re(\sigma) = \alpha > 1$.

Numerical approximations of $\gamma_\ell(a)$ have been recently given in [2, 6, 8]. In the last reference, Padé approximation of the remainder term of a series giving Stieltjes constant provides a new approximation of this constant.

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The goal of the current paper is to write new series for $\gamma_\ell(a)$ whose terms are the derivatives of the zeta function at integers. These results will generalize the formula for Euler constant proved in [5].

In [7], using the Padé approximation $[n, m]_{e^x} = \frac{\sum_{j=0}^n \frac{(-n)_j}{(-m-n)_j} \frac{x^j}{j!}}{\sum_{j=0}^m \frac{(-m)_j}{(-m-n)_j} \frac{(-x)^j}{j!}}$ of the exponential function in the following integral representation of $\zeta(\sigma, a)$,

$$\zeta(\sigma, a) = \frac{1}{\Gamma(\sigma)} \int_0^\infty \frac{x^{\sigma-1} e^{-ax}}{1 - e^{-x}} dx \quad (2)$$

we proved the following result:

For $n, m \in \mathbb{N}, \sigma \in \mathbb{C} \setminus D_{n,m}$ where $D_{n,m} := \{-n - m, 1 - \max(n, m), \dots, -1, 0, 1\}$, for $(p, a) \in \mathbb{C}^2$, let us define the quantity $A_{(n,m)}^{(p)}(\sigma, a)$ by

$$A_{(n,m)}^{(p)}(\sigma, a) := \sum_{j=0}^m \frac{(-m)_j}{(-n-m)_j} p^j \frac{(\sigma)_j}{j!} \zeta(\sigma + j, a + p) - \sum_{j=0}^n \frac{(-n)_j}{(-n-m)_j} (-p)^j \frac{(\sigma)_j}{j!} \zeta(\sigma + j, a). \quad (3)$$

Theorem 1. *If $\Re(a) > 0$ and $\Re(a + p) > 0$, then*

$$A_{(n,m)}^{(p)}(\sigma, a) = (-1)^{n+1} p^{m+n+1} \frac{(\sigma)_{m+n+1}}{(n+m)!} \int_0^1 x^m (1-x)^n \zeta(\sigma + m + n + 1, a + px) dx$$

where $(\sigma)_j := \sigma(\sigma + 1) \cdots (\sigma + j - 1) = \frac{\Gamma(\sigma + j)}{\Gamma(\sigma)}$ is the Pochhammer symbol.

Remark 1. The conditions $\Re(a) > 0$ and $\Re(a + p) > 0$ are not restrictive since $\zeta(\sigma, a) = \zeta(\sigma, a + k) + \sum_{i=0}^{k-1} (i + a)^{-\sigma} > 0$ and $\Re(a + k) > 0$ for suitable integer k .

The convergence of $A_{(n,m)}^{(p)}$ when n or m tends to infinity has been proved by the following Lemma.

Lemma 1. *If $|p| \leq |a|$, $|p| \leq |a + p|$, $\Re(a) > 0$ and $\Re(a + p) > 0$, then $\forall \sigma \in \mathbb{C} \setminus \mathbb{Z}^-, \sigma \neq 1$*

$$\left| \int_0^1 x^m (1-x)^n \zeta(\sigma + m + n + 1, a + px) dx \right| \leq C \frac{1}{|a + p|^m} \frac{1}{|a|^n}$$

where C is some constant independent of m and n .

So, under conditions on a and p , the previous lemma shows that the quantity $A_{(n,m)}^{(p)}(\sigma, a)$ tends to 0 when m or n tends to infinity.

In this paper, we use Theorem 1 to construct formulas for Stieltjes constants. To do that, we will use the following expression for Stieltjes constant

$$\gamma_\ell(a) = (-1)^\ell \left(\frac{d}{d\sigma} \right)^\ell \left(\zeta(\sigma, a) - \frac{1}{\sigma - 1} \right) \Big|_{\sigma=1}.$$

2. RESULTS

In this section, we prove a general formula for Stieltjes constant which depends on two parameters: n is the degree of the numerator and m the degree of the denominator of the Padé approximant used to approximate the function e^{-x} in (2).

We denote by $\zeta^{(r,0)}(j, a)$ the r -th derivative of the function $\zeta(\sigma, a)$ with respect to the variable σ , computed at $\sigma = j$.

Theorem 2. *Suppose that $\Re(a) > 0$ and $|a| \geq 1$. Then*

$$\begin{aligned} (-1)^\ell \gamma_\ell(a) &= \sum_{j=2}^{\max(m,n)} \frac{(-m)_j - (-1)^j (-n)_j}{(-n-m)_j} \frac{(-1)^j}{j!} \sum_{k=0}^{\ell} s(j, k+1) \frac{\ell!}{(\ell-k)!} (-1)^k \zeta^{(\ell-k,0)}(j, a) + \\ &\quad \sum_{k=0}^{\ell} \frac{\ell!}{(\ell-k)!} \ln^{\ell-k} a \sum_{j=1}^m \frac{(-m)_j}{(-n-m)_j} \frac{(-1)^{j+\ell+1} a^{-j}}{j!} s(j, k+1) \\ &\quad + \frac{\ln^{\ell+1} a}{\ell+1} (-1)^{\ell+1} + R_{\ell,m,n,a}, \end{aligned} \quad (4)$$

where $s(j, k)$ are the Stirling numbers of the first kind defined by $(x)_n = \sum_{k=0}^n (-1)^{n-k} s(n, k) x^k$ and the remainder term is

$$R_{\ell,m,n,a} = \frac{(-1)^{m+1}}{(m+n)!} \sum_{k=0}^{\ell} s(m+n+1, k+1) \frac{\ell!}{(\ell-k)!} (-1)^k \int_0^1 x^m (1-x)^n \zeta^{(\ell-k,0)}(m+n+1, a+x) dx. \quad (5)$$

The Stirling numbers satisfies also some known properties used in this paper:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} s(n, k) = \frac{(\ln(1+x))^k}{k!}, \quad |x| < 1, \quad (6)$$

$$s(n, 1) = (-1)^{n-1} (n-1)!, \quad (7)$$

$$s(n, 2) = (-1)^n (n-1)! H_{n-1}, \quad (8)$$

where $H_j = 1 + \frac{1}{2} + \dots + \frac{1}{j} = \int_0^1 \frac{1-t^j}{1-t} dt$ is the Harmonic number.

Particular case. $\ell = 0$

$$\begin{aligned} \gamma(a) &= - \sum_{j=2}^{\max(m,n)} \frac{(-m)_j - (-1)^j (-n)_j}{(-n-m)_j} \frac{1}{j} \zeta(j, a) + \sum_{j=1}^m \frac{(-m)_j}{(-n-m)_j} \frac{a^{-j}}{j} - \\ &\quad - \ln a + R_{0,m,n,a}, \end{aligned} \quad (9)$$

where

$$R_{0,m,n,a} = (-1)^n \int_0^1 x^m (1-x)^n \zeta(m+n+1, a+x) dx.$$

For $a = 1$, formula (9) becomes

$$\gamma = - \sum_{j=2}^{\max(m,n)} \frac{(-m)_j - (-1)^j (-n)_j}{(-n-m)_j} \frac{1}{j} \zeta(j) + \sum_{j=1}^m \frac{(-m)_j}{(-n-m)_j} \frac{1}{j} + \varepsilon_{n,m}, \quad (10)$$

where $\varepsilon_{n,m} = (-1)^n \int_0^1 x^m (1-x)^n \zeta(m+n+1, 1+x) dx$.

The previous formula has been proved in [5, p. 512].

3. PROOF OF THEOREM 2

First, for $p = 1$, we divide the relation (3) by s , we replace σ by $\sigma - 1$ and we compute the ℓ -th derivative at $\sigma = 1$.

$$\frac{A_{(n,m)}^{(1)}(\sigma-1, a)}{\sigma-1} = \sum_{j=0}^m \frac{(-m)_j}{(-n-m)_j} \frac{(\sigma)_{j-1}}{j!} \zeta(\sigma-1+j, a+1) - \sum_{j=0}^n \frac{(-n)_j (-1)^j}{(-n-m)_j} \frac{(\sigma)_{j-1}}{j!} \zeta(\sigma-1+j, a) \quad (11)$$

$$= (-1)^{n+1} \frac{(\sigma)_{m+n}}{(n+m)!} \int_0^1 x^m (1-x)^n \zeta(\sigma+m+n, a+x) dx. \quad (12)$$

In the relation (11), the term for $j = 0$ is $\frac{1}{\sigma-1} \zeta(\sigma-1, a+1) - \frac{1}{\sigma-1} \zeta(\sigma-1, a) = \frac{-1}{\sigma-1} a^{-\sigma+1}$. For $j = 1$, it is $\frac{-m}{-n-m} \zeta(\sigma, a+1) - \frac{n}{-n-m} \zeta(\sigma, a) = \zeta(\sigma, a) - \frac{m}{n+m} a^{-\sigma}$.

Thus

$$\begin{aligned} \left(\frac{d}{d\sigma} \right)^\ell \frac{A_{(n,m)}^{(1)}(\sigma-1, a)}{\sigma-1} \Big|_{\sigma=1} &= \left(\frac{d}{d\sigma} \right)^\ell \left(\zeta(\sigma, a) - \frac{a^{-\sigma+1}}{\sigma-1} - \frac{m}{n+m} a^{-\sigma} \right) \Big|_{\sigma=1} \\ &+ \sum_{j=2}^m \frac{(-m)_j}{(-n-m)_j} \frac{1}{j!} \left(\frac{d}{d\sigma} \right)^\ell (\sigma)_{j-1} \zeta(\sigma-1+j, a+1) \Big|_{\sigma=1} \\ &- \sum_{j=2}^n \frac{(-n)_j}{(-n-m)_j} \frac{(-1)^j}{j!} \left(\frac{d}{d\sigma} \right)^\ell (\sigma)_{j-1} \zeta(\sigma-1+j, a) \Big|_{\sigma=1}. \end{aligned}$$

Using the relation $\zeta(\sigma, a+1) = \zeta(\sigma, a) - a^{-\sigma}$, we find

$$\begin{aligned} \left(\frac{d}{d\sigma} \right)^\ell \frac{A_{(n,m)}^{(1)}(\sigma-1, a)}{\sigma-1} \Big|_{\sigma=1} &= \left(\frac{d}{d\sigma} \right)^\ell \left(\zeta(\sigma, a) - \frac{a^{-\sigma+1}}{\sigma-1} \right) \Big|_{\sigma=1} \\ &+ \sum_{j=2}^m \frac{(-m)_j - (-n)_j (-1)^j}{(-n-m)_j} \frac{1}{j!} \left(\frac{d}{d\sigma} \right)^\ell (\sigma)_{j-1} \zeta(\sigma-1+j, a) \Big|_{\sigma=1} \\ &- \sum_{j=1}^m \frac{(-m)_j}{(-n-m)_j} \frac{1}{j!} \left(\frac{d}{d\sigma} \right)^\ell (\sigma)_{j-1} a^{-\sigma-j+1} \Big|_{\sigma=1}. \end{aligned}$$

Now, we use the formula [4, Lemma 1]

$$\left(\frac{d}{d\sigma}\right)^\ell (\sigma)_{j-1} \Big|_{\sigma=1} = s(j, \ell+1) \ell! (-1)^{j-1+\ell}, j \geq 1.$$

Then, with the product rule, we obtain

$$\left(\frac{d}{d\sigma}\right)^\ell (\sigma)_{j-1} \zeta(\sigma-1+j, a) \Big|_{\sigma=1} = \sum_{k=0}^{\ell} s(j, k+1) \frac{\ell!}{(\ell-k)!} (-1)^{j-1+k} \zeta^{(\ell-k)}(j, a), j \geq 2,$$

and

$$\left(\frac{d}{d\sigma}\right)^\ell (\sigma)_{j-1} a^{-\sigma-j+1} \Big|_{\sigma=1} = \sum_{k=0}^{\ell} \frac{\ell!}{(\ell-k)!} (\ln a)^{\ell-k} (-1)^{j+\ell+1} a^{-j} s(j, k+1), j \geq 1,$$

$$\begin{aligned} \left(\frac{d}{d\sigma}\right)^\ell \left(\zeta(\sigma, a) - \frac{a^{-\sigma+1}}{\sigma-1}\right) \Big|_{\sigma=1} &= \left(\frac{d}{d\sigma}\right)^\ell \left(\zeta(\sigma, a) - \frac{1}{\sigma-1} + \frac{1-a^{-\sigma+1}}{\sigma-1}\right) \Big|_{\sigma=1} \\ &= (-1)^\ell \gamma_\ell(a) + (-1)^\ell \frac{\ln^{\ell+1} a}{\ell+1}. \end{aligned}$$

It arises

$$\begin{aligned} (-1)^\ell \gamma_\ell(a) &= -\frac{\ln^{\ell+1} a}{\ell+1} (-1)^\ell + \sum_{k=0}^{\ell} \frac{\ell!}{(\ell-k)!} \ln^{\ell-k} a \sum_{j=1}^m \frac{(-m)_j}{(-n-m)_j} \frac{(-1)^{j+\ell+1} a^{-j}}{j!} s(j, k+1) \\ &\quad + \sum_{j=2}^{\max(m,n)} \frac{(-m)_j - (-1)^j (-n)_j}{(-n-m)_j} \frac{(-1)^{j+1}}{j!} \sum_{k=0}^{\ell} s(j, k+1) \frac{\ell!}{(\ell-k)!} (-1)^k \zeta^{(\ell-k,0)}(j, a) + R_{\ell,m,n,a} \end{aligned}$$

where

$$R_{\ell,m,n,a} = \frac{(-1)^{m+1}}{(m+n)!} \sum_{k=0}^{\ell} s(m+n+1, k+1) \frac{\ell!}{(\ell-k)!} (-1)^k \int_0^1 x^m (1-x)^n \zeta^{(\ell-k,0)}(m+n+1, a+x). \quad (13)$$

The derivation of $\frac{A_{(n,m)}^{(1)}(\sigma-1, a)}{\sigma-1}$ with respect to σ is justified by Lemma 3 since the function $\phi(x)$ is integrable.

In the following, we slightly modify Lemma 1 for $p = 1$.

Lemma 2. *Suppose that $\Re(a) > 0$ and $|a| \geq 1$. Then $\forall k \in [0, \ell], \forall m, n$ such that $m+n \geq l+1$,*

$$\left| \int_0^1 x^m (1-x)^n \zeta(m+n+1+k-\ell, a+x) dx \right| \leq K \frac{1}{|a+1|^m} \frac{1}{|a|^n},$$

where K is some constant independent of m and n .

Proof.

$$\begin{aligned} \left| \int_0^1 x^m (1-x)^n \zeta(m+n+1+k-\ell, a+x) dx \right| &\leq \int_0^1 x^m (1-x)^n |\zeta(m+n+1+k-\ell, a+x)| dx \\ &\leq \int_0^1 \left| \frac{x}{a+x} \right|^m \left| \frac{1-x}{a+x} \right|^n |a+x|^{l-k-1} |\zeta(m+n+1+k-\ell, a+x)| |a+x|^{m+n+1+k-l} dx. \end{aligned}$$

In [7], we proved that for $\Re(z) \geq 1$, $|\zeta(z, a+x)(a+x)^z|$ is bounded by some constant C independent of z and that $\max_{0 \leq x \leq 1} \left| \frac{x}{a+x} \right| = \frac{1}{|a+1|}$ and $\max_{0 \leq x \leq 1} \left| \frac{1-x}{a+x} \right| = \frac{1}{|a|}$. This proves the Lemma, where $K = C \int_0^1 |a+x|^{l-k-1} dx$. \square

In order to prove the convergence to 0 of the remainder term $R_{l,m,n,a}$ of our principal formula (4), we need a preliminary technical result.

Lemma 3. *Suppose that $r \in \mathbb{N}$, $a \in \mathbb{C}$, $\Re(a) > 0$ and $|a| \geq 1$, $j > r + 1$.*

Then $\forall x \geq 0$, $|a+x| \geq 1$, and

$$|\zeta^{(r,0)}(j, a+x)| \leq \sum_{k \geq 0} \frac{((\ln |k+a+x|)^2 + \pi^2/4)^{r/2}}{|k+a+x|^j} =: \phi(x) \quad (14)$$

$$\phi(x) \leq 2^r \zeta(j-r, \Re(a)+x) \quad (15)$$

Proof.

$$\begin{aligned} |\zeta^{(r,0)}(j, a+x)| &= \left| \sum_{k \geq 0} (-1)^r \frac{(\ln(k+a+x))^r}{(k+a+x)^j} \right| \leq \sum_{k \geq 0} \frac{|s(k+a+x)|^r}{|k+a+x|^j} \\ &= \sum_{k \geq 0} \frac{|\ln |k+a+x| + i \operatorname{Arg}(k+a+x)|^r}{|k+a+x|^j} \\ &= \sum_{k \geq 0} \frac{(\ln |k+a+x|)^2 + (\operatorname{Arg}(k+a+x))^2)^{r/2}}{|k+a+x|^j} \\ &\leq \sum_{k \geq 0} \frac{(\ln |k+a+x|)^2 + (\pi/2)^2)^{r/2}}{|k+a+x|^j} \leq \sum_{k \geq 0} \frac{(|k+a+x|^2 + 3|k+a+x|^2)^{r/2}}{|k+a+x|^j} \\ &\leq 2^r \sum_{k \geq 0} \frac{|k+a+x|^r}{|k+a+x|^j} = 2^r \sum_{k \geq 0} \frac{1}{|k+a+x|^{j-r}} \\ &\leq 2^r \sum_{k \geq 0} \frac{1}{(k + \Re(a) + x)^{j-r}} = 2^r \zeta(j-r, \Re(a)+x) \end{aligned}$$

\square

Theorem 3. *If $\Re(a) \geq 1$ then*

$$\forall n \in \mathbb{N}, \lim_{m \rightarrow \infty} R_{\ell, m, n, a} = 0,$$

$$\forall m \in \mathbb{N}, \lim_{n \rightarrow \infty} R_{\ell, m, n, a} = 0.$$

Proof. From (13), we can write

$$|R_{\ell, m, n, a}| \leq \frac{1}{(m+n)!} \sum_{k=0}^{\ell} |s(m+n+1, k+1)| \frac{\ell!}{(\ell-k)!} \int_0^1 x^m (1-x)^n |\zeta^{(\ell-k, 0)}(m+n+1, a+x)| dx$$

□

Adell in [1], proved some explicit upper bounds for the Stirling numbers of the first kind. We will use the following formula:

$$\text{for } l = 1, \dots, j-1, \quad |s(j+1, \ell+1)| \leq \frac{j!}{\ell!} (\ln j)^\ell \left(1 + \frac{\ell}{\ln j}\right).$$

$$\begin{aligned} |R_{\ell, m, n, a}| &\leq \sum_{k=0}^{\ell} (\ln(m+n))^k \left(1 + \frac{k}{\ln(m+n)}\right) \frac{1}{(\ell-k)!} \int_0^1 x^m (1-x)^n |\zeta^{(\ell-k, 0)}(m+n+1, a+x)| dx \\ &\leq \left(1 + \frac{\ell}{\ln(m+n)}\right) (\ln(m+n))^\ell \sum_{k=0}^{\ell} 2^{\ell-k} \int_0^1 x^m (1-x)^n \zeta(m+n+1-\ell+k, \Re(a)+x) \end{aligned}$$

(Lemma 3)

$$\leq C \left(1 + \frac{\ell}{\ln(m+n)}\right) (\ln(m+n))^{\ell} 2^{\ell+1} \frac{1}{(\Re(a)+1)^m} \frac{1}{\Re(a)^n}. \quad (16)$$

The second inequality is valid since m and/or n tends to infinity and thus the parameter $m+n+1+k-\ell$ is greater than 1.

Thus, if $\Re(a) > 1$, the limit of the remainder $R_{\ell, m, n, a}$ is 0 when m or n tend to infinity and we have

$$\lim_{m \rightarrow \infty} |R_{\ell, m, n, a}|^{1/m} \leq \frac{1}{\Re(a)+1},$$

$$\lim_{n \rightarrow \infty} |R_{\ell, m, n, a}|^{1/n} \leq \frac{1}{\Re(a)}.$$

If $\Re(a) = 1$, then $\lim_{m \rightarrow \infty} |R_{\ell, m, n, a}|^{1/m} \leq 1/2$.

Now, if $\Re(a) = 1$ and n tends to infinity, we have to bound the integral term of right hand side of (16) as following.

$$\begin{aligned} \int_0^1 x^m (1-x)^n \zeta(m+n+1+k-\ell, 1+x) dx &\leq \int_0^1 (1-x)^n \zeta(m+n+1+k-\ell, 1+x) dx \\ &\leq \int_0^1 \frac{(1-x)^n}{(1+x)^{m+n+1+k-\ell}} (1+x)^{m+n+1+k-\ell} \zeta(m+n+1+k-\ell, 1+x) dx \end{aligned}$$

$$\begin{aligned} &\leq \int_0^1 \frac{(1-x)^n}{(1+x)^n} (1+x)^{m+n+1+k-\ell} \zeta(m+n+1+k-\ell, 1+x) dx \\ &\leq C \int_0^1 \frac{(1-x)^n}{(1+x)^n} dx \leq \frac{C}{2n}. \end{aligned}$$

(see Lemma 2).

We can conclude that, when $\Re(a) = 1$ the remainder term tends to 0 when n tends to infinity.

Remark

To accelerate the computation, we can use the following relation (for p integer):

$$\gamma_\ell(a) = \gamma_\ell(a+p) + \sum_{k=0}^{p-1} \frac{\ln^\ell(a+k)}{a+k}.$$

It leads to

$$\begin{aligned} (-1)^\ell \gamma_\ell(a) &= (-1)^\ell \gamma_\ell(a+p) + (-1)^\ell \sum_{k=0}^{p-1} \frac{\ln^\ell(a+k)}{a+k} \\ &= (-1)^\ell \sum_{k=0}^{p-1} \frac{\ln^\ell(a+k)}{a+k} + \\ &\quad \sum_{j=2}^{\max(m,n)} \frac{(-m)_j - (-1)^j (-n)_j}{(-n-m)_j} \frac{(-1)^{j+1}}{j!} \sum_{k=0}^{\ell} S(j, k+1) \frac{\ell!}{(\ell-k)!} (-1)^k \zeta^{(\ell-k,0)}(j, a+p) + \\ &\quad \sum_{k=0}^{\ell} \frac{\ell!}{(\ell-k)!} \ln^{\ell-k}(a+p) \sum_{j=1}^m \frac{(-m)_j}{(-n-m)_j} \frac{(-1)^{j+\ell+1} (a+p)^{-j}}{j!} S(j, k+1) + R_{\ell,m,n,a+p}, \end{aligned}$$

where the remainder terms $R_{\ell,m,n,a+p}$ converges to 0 as $\frac{1}{\Re(a+p+1)^m \Re(a+p)^n}$

4. PARTICULAR CASES

In this section, we consider all the possible values for the four parameters m, n, a .

4.1. $m \in \mathbb{N}, n = 0, \Re(a) \geq 1$. After simplification, we get

$$\begin{aligned}
(-1)^\ell \gamma_\ell(a) &= \sum_{j=2}^m \frac{(-1)^j}{j!} \sum_{k=0}^{\ell} s(j, k+1) \frac{\ell!}{(\ell-k)!} (-1)^k \zeta^{(\ell-k,0)}(j, a) \\
&+ \sum_{k=0}^{\ell} \frac{\ell!}{(\ell-k)!} \ln^{\ell-k} a \sum_{j=1}^m \frac{(-1)^{j+\ell+1} a^{-j}}{j!} s(j, k+1) \\
&+ \frac{\ln^{\ell+1} a}{\ell+1} (-1)^{\ell+1} + R_{\ell, m, 0, a}.
\end{aligned} \tag{17}$$

If m tends to infinity, it arises (see Theorem 3)

$$\begin{aligned}
(-1)^\ell \gamma_\ell(a) &= \sum_{j=2}^{\infty} \frac{(-1)^j}{j!} \sum_{k=0}^{\ell} s(j, k+1) \frac{\ell!}{(\ell-k)!} (-1)^k \zeta^{(\ell-k,0)}(j, a) \\
&+ \sum_{k=0}^{\ell} \frac{\ell!}{(\ell-k)!} \ln^{\ell-k} a \sum_{j=1}^{\infty} \frac{(-1)^{j+\ell+1} a^{-j}}{j!} s(j, k+1) \\
&+ \frac{\ln^{\ell+1} a}{\ell+1} (-1)^{\ell+1}.
\end{aligned} \tag{18}$$

After simplification (using relation (6)) leads to

$$(-1)^\ell \gamma_\ell(a) = \frac{\ln^{\ell+1}(a-1)}{\ell+1} (-1)^{\ell+1} + \sum_{j=2}^{\infty} \frac{(-1)^j}{j!} \sum_{k=0}^{\ell} s(j, k+1) \frac{\ell!}{(\ell-k)!} (-1)^k \zeta^{(\ell-k,0)}(j, a).$$

4.2. $n \in \mathbb{N}, m = 0, \Re(a) \geq 1$. After simplification, we get

$$(-1)^\ell \gamma_\ell(a) = \sum_{j=2}^n \frac{1}{j!} \sum_{k=0}^{\ell} s(j, k+1) \frac{\ell!}{(\ell-k)!} (-1)^k \zeta^{(\ell-k,0)}(j, a) + \frac{\ln^{\ell+1} a}{\ell+1} (-1)^{\ell+1} + R_{\ell, 0, n, a}.$$

If n tends to infinity, it arises (see Theorem 3)

$$(-1)^\ell \gamma_\ell(a) = \sum_{j=2}^{\infty} \frac{1}{j!} \sum_{k=0}^{\ell} s(j, k+1) \frac{\ell!}{(\ell-k)!} (-1)^k \zeta^{(\ell-k,0)}(j, a) + \frac{\ln^{\ell+1} a}{\ell+1} (-1)^{\ell+1}.$$

5. CASE $\Re(a) > 0, |a| \geq 1, m = \lambda n, m, n \rightarrow \infty$

Now, we consider the case $m = \lambda n$, when n tends to infinity.

Of course, the general expression of Theorem 2 is true and we can derive the following series

$$\begin{aligned} (-1)^\ell \gamma_\ell(a) &= -\frac{\ln^{\ell+1} a}{\ell+1} (-1)^\ell + \frac{\lambda}{\lambda+1} \frac{\ln^\ell a}{a} (-1)^\ell \\ &\quad - \sum_{j=2}^{\infty} \frac{(\lambda)^j - (-1)^j}{(\lambda+1)^j} \frac{(-1)^{j+1}}{j!} \sum_{k=0}^{\ell} s(j, k+1) \frac{\ell!}{(\ell-k)!} (-1)^k \zeta^{(\ell-k,0)}(j, a) \\ &\quad + \sum_{k=0}^{\ell} (-1)^{\ell+1} \frac{\ell!}{(\ell-k)!} \ln^{\ell-k}(a) \sum_{j=2}^{\infty} \left(\frac{-\lambda/a}{\lambda+1} \right)^j \frac{1}{j!} s(j, k+1). \end{aligned}$$

Using the relation (6) and for $\left| \frac{\lambda/a}{\lambda+1} \right| < 1$, it can be simplified as

$$(-1)^\ell \gamma_\ell(a) = (-1)^{\ell+1} \frac{\ln^{\ell+1} \left(a - \frac{\lambda}{\lambda+1} \right)}{\ell+1} - \sum_{k=0}^{\ell} \frac{\ell! (-1)^k}{(\ell-k)!} \left[\sum_{j=2}^{\infty} \frac{(\lambda)^j - (-1)^j}{(\lambda+1)^j} \frac{(-1)^{j+1}}{j!} s(j, k+1) \zeta^{(\ell-k,0)}(j, a) \right]. \quad (19)$$

One may consider (19) as an identity between two holomorphic functions in the variable λ . Since this formula is true for $\lambda \in \mathbb{Q}$ such that $\left| \frac{\lambda}{\lambda+1} \right| < |a|$, it is obviously true for every $\lambda \in \mathbb{C}, \left| \frac{\lambda}{\lambda+1} \right| < |a|$.

Now, suppose that a is real greater than 1 and we choose $\lambda = i$.

Equating real and imaginary parts of (19) we get

$$\begin{aligned} (-1)^\ell \gamma_\ell(a) &= (-1)^{\ell+1} \Re \left(\frac{\ln^{\ell+1} (a - 1/2 - i/2)}{\ell+1} \right) \\ &\quad - \sum_{k=0}^{\ell} \frac{\ell! (-1)^k}{(\ell-k)!} \left[\begin{aligned} &\sum_{q=1}^{\infty} (-1)^q 2^{-2q} \frac{1}{(4q+1)!} s(4q+1, k+1) \zeta^{(\ell-k,0)}(4q+1, a) \\ &- \sum_{q=0}^{\infty} (-1)^q 2^{-2q-1} \frac{1}{(4q+3)!} s(4q+3, k+1) \zeta^{(\ell-k,0)}(4q+3, a) \end{aligned} \right] \end{aligned}$$

and the following identity between Hurwitz- ζ functions:

$$\sum_{k=0}^{\ell} \frac{\ell!}{(\ell-k)!} (-1)^k \sum_{j \equiv 2(4)}^{\infty} (-1/4)^{\frac{j-2}{4}} \frac{(-1)^{j-1}}{j!} s(j, k+1) \zeta^{(\ell-k,0)}(j, a) = \Im \left((-1)^{\ell+1} \frac{\ln \left(a - \frac{i+1}{2} \right)^{\ell+1}}{\ell+1} \right), \quad (20)$$

or more simply

$$\sum_{k=0}^{\ell} \frac{\ell!}{(\ell-k)!} (-1)^k \sum_{r=0}^{\infty} \frac{(-1/4)^r}{(4r+2)!} s(4r+2, k+1) \zeta^{(\ell-k,0)}(4r+2, a) = \Im \left((-1)^\ell \frac{\ln \left(a - \frac{i+1}{2} \right)^{\ell+1}}{\ell+1} \right). \quad (21)$$

6. PARTICULAR CASE: $\ell = 1$

In this section, we consider the first Stieltjes constant γ_1 . We will show a new formula of this constant in terms of the first derivatives of the ζ -functions.

Proposition.

$$\gamma_1 = \sum_{j=1}^{\infty} \frac{\zeta'(2j+1)}{2j+1}.$$

Proof. We use the formula (19) with $a = 1$, $l = 1$ and $\lambda = 0$.

$$\gamma_1 = \sum_{j=2}^{\infty} \frac{1}{j!} (s(j, 1)\zeta'(j) - s(j, 2)\zeta(j)) \quad (22)$$

The formula (19) with $a = 2$, $l = 1$ and $\lambda \rightarrow \infty$.

$$\gamma_1(2) = \gamma_1 = \sum_{j=2}^{\infty} \frac{1}{j!} (-1)^{j-1} (s(j, 1)\zeta'(j, 2) - s(j, 2)\zeta(j, 2)) \quad (23)$$

$$= \sum_{j=2}^{\infty} \frac{1}{j!} (-1)^{j-1} (s(j, 1)\zeta'(j) - s(j, 2)\zeta(j, 2)) \quad (24)$$

Using the expression of the first Stirling numbers (7, 8), the two expression of γ_1 become

$$\gamma_1 = \sum_{j=2}^{\infty} \frac{(-1)^{j-1}}{j} (\zeta'(j) + H_{j-1}\zeta(j)) \quad (25)$$

$$\gamma_1 = \sum_{j=2}^{\infty} \frac{1}{j} (\zeta'(j) + H_{j-1}\zeta(j, 2)) \quad (26)$$

In the sequel, we will show that

$$\sum_{j=2}^{\infty} \frac{H_{j-1}}{j} (\zeta(j, 2) + (-1)^{j-1}\zeta(j)) = 0$$

and the proposition will be proved.

$$\sum_{j=2}^{\infty} \frac{H_{j-1}}{j} (\zeta(j, 2) + (-1)^{j-1}\zeta(j)) = \int_0^1 \sum_{j=2}^{\infty} \frac{1}{j} (\zeta(j, 2) + (-1)^{j-1}\zeta(j)) \frac{1-t^{j-1}}{1-t} dt.$$

The permutation is valid since the convergence of the series is uniform on $[0, 1]$. Using the formula (<https://dlmf.nist.gov/5.7.E3>)

$$\ln \Gamma(1+z) = -\ln(1+z) + z(1-\gamma) + \sum_{k=2}^{\infty} (-1)^k \zeta(k, 2) \frac{z^k}{k}, \quad |z| < 2,$$

it is easy to prove that

$$\begin{aligned} \sum_{j \geq 2} \frac{1}{j} (\zeta(j, 2) + (-1)^{j-1} \zeta(j)) \frac{1-t^{j-1}}{1-t} &= \sum_{j \geq 2} \frac{1}{j} \zeta(j, 2) \frac{1-t^{j-1}}{1-t} + \sum_{j \geq 2} \frac{(-1)^{j-1}}{j} \zeta(j) \frac{1-t^{j-1}}{1-t} \\ &= -\frac{\ln \Gamma(2-t)}{t(1-t)} + \frac{\ln \Gamma(1+t)}{t(1-t)} =: \phi(t). \end{aligned}$$

This function ϕ is symmetric with respect to the abscissa $t = 1/2$, so its integral between 0 and 1 is zero and the proposition is proved. \square

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