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# PARAMETERS IDENTIFICATION IN A SALTWATER INTRUSION PROBLEM \*

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**Abstract** This article is devoted to the identification, from observations or field measurements, of the hydraulic conductivity  $K$  for the saltwater intrusion problem in confined aquifer. The involved PDE model is a coupled system of nonlinear parabolic-elliptic equations completed by boundary and initial conditions. The main unknowns are the saltwater/ freshwater interface depth and the freshwater hydraulic head. The inverse problem is formulated as an optimization problem where the cost function is a least square functional measuring the discrepancy between experimental interface depths and those provided by the model. Considering the exact problem as a constraint for the optimization problem and introducing the Lagrangian associated with the cost function, we prove that the optimality system has at least one solution.

**Key words** strongly coupled system, nonlinear parabolic equations, parameters identification, optimization problem, seawater intrusion.

**2010 MR Subject Classification** 35K51; 35K55; 49K20

## 1 Introduction

In order to get an optimal exploitation of freshwater and then to control seawater intrusion in coastal aquifers, we need to develop efficient and accurate models for simulating the transport of saltwater front in coastal aquifer. An essential step is thus to identify the parameters involved in the model.

In this article, we focus on the identification of physical parameters such as the hydraulic conductivity  $K$  and the porosity  $\phi$ . The estimation of these parameters are based on observations or field measurements made on hydraulic heads and on the depth of the freshwa-

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ter/saltwater interface. We note that concretely, we only have specific observations (in space and in time) corresponding to the number of monitoring wells.

We emphasize also that the parameters identification problematic was often addressed as part of the underground hydraulic (cf. [7, 12, 13, 15, 18, 22]), but more rarely with regard to saltwater intrusion phenomena. The first studies done by Sun et Yeh (cf. [19]) develop the case of resolutions of inverse problem for coupled systems and provide, in particular in the case of saltwater intrusion, the associated adjoint system corresponding to the steady system.

Moreover, we note that existing studies mainly concern numerical resolutions of these inverse problems. In particular, we must mention the study done in [20], where the authors focus on the numerical resolutions of the hydraulic conductivity identification in the case of confined aquifers. The system of optimality is discretized by a finite element method; For solving the optimization problem, the authors used the C++ finite element library Rheolef and a C/C++ program based on the advanced optimization package (TAO) that implements the BLMVM algorithm.

In [21], the authors propose a theoretical study by proving the existence of an optimal control and they give the necessary optimality conditions in the case of a steady saltwater/freshwater interface. But the seawater intrusion phenomenon is often transient and the study of sensitivity proposed in [17] shows that the form of freshwater/saltwater interface depends mainly on the hydraulic conductivity, the other parameters such as especially porosity, have an impact on the time taken to reach the steady state, which is, why we must consider the unsteady model to identify simultaneously these two parameters.

The current article is a generalization of [21] to the unsteady case. The inverse problem then reduces in an optimization problem, where the cost function is a least square functional measuring the discrepancy between experimental interface depth and the freshwater hydraulic head and those given by the model. Clearly, the time dependance of the variables is one of the difficulties of this study. Then, by considering the exact problem as a constraint for the optimization problem, we introduce the Lagrangian associated with the cost function. In the field envisaged here, the effects due to porosity variations are negligible compared with those due to the density contrasts. We thus assume that porosity is constant in the aquifer and we focus only on the hydraulic conductivity identification.

Before to detail the mathematical analysis of this optimization problem, let us describe briefly the derivation of the model. We refer to the textbooks [4–6] for general informations about seawater intrusion problems. The basis of the modeling is the mass conservation law for each species (fresh and salt water) combined with the classical Darcy law for porous media. In the present work we essentially have chosen to adopt the simplicity of a sharp interface approach. Indeed, observations show that, near the shoreline, fresh and salty underground water tend to separate into two distinct layers. It was the motivation for the derivation of seawater intrusion models treating salt- and freshwater as immiscible fluids. Nevertheless the explicit tracking of the interfaces remains unworkable to implement without further assumptions. An additional assumption, the so-called Dupuit approximation, consists in considering that the hydraulic head is constant along each vertical direction. It allows to assume the existence of a smooth sharp interface. Classical sharp interface models are then obtained by vertical inte-

gration based on the assumption that no mass transfer occurs between the fresh and the salty area (see *e.g.* [5] and even the Ghyben–Herzberg static approximation). This class of models allows direct tracking of the salt front. Nevertheless the conservative form of the equations is perturbed by the upscaling procedure. In particular the maximum principle does not apply. Following [9], we can mix the latter abrupt interface approach with a phase field approach (here an Allen–Cahn type model in fluid–fluid context see *e.g.* [1, 2, 8]) for re-including the existence of a diffuse interface between fresh and saltwater where mass exchanges occur. We thus combine the advantage of respecting the physics of the problem and that of the computational efficiency. From a theoretical point of view, an advantage resulting from the addition of diffuse area compared to the sharp interface approximation is that the system now has a parabolic structure, so it is not necessary to introduce viscous terms in a preliminary fixed point for treating degeneracy as in the case of sharp interface approach. (cf. [9]). The second advantage is that we can prove the uniqueness of the solution by establishing a regularity result on the gradient of the solution. More precisely we generalize to the quasilinear case, the regularity result given by Meyers ([16]) in the elliptic case and extended to the parabolic case by A. Bensoussan, JL Lions and G. Papanicolaou, for any elliptic operator  $A = -\sum_{i,j=1}^n \partial_j a_{ij}(x) \partial_i$  (see [3]). The results assume that the operator  $A$  satisfies an uniform ellipticity assumption and that its coefficients are functions of  $L^\infty(\Omega)$ . The hypothesis on  $A$  ensure the existence of an exponent  $r(A) > 2$  such that the gradient of the solution of the elliptic equation (resp. of the parabolic equation) belongs to the space  $L^r(\Omega)$  (resp.  $L^r(\Omega_T)$ ). This additional regularity combined with the Gagliardo–Nirenberg inequality can handle the nonlinearity of the system in the proof of uniqueness. For these reasons, we limit ourselves in this paper to the case of diffuse interface approach.

The first difficulty of the work consists in choosing appropriately the set of all eligible parameters. Taking this set in space  $L^\infty(\Omega)$  ensures the existence and uniqueness of the solution of initial problem but does not provide the existence of the optimal control problem. Also take this set in  $H^1(\Omega)$  is a too restrictive hypothesis that must be weakened. Therefore we introduce the set of admissible parameters defined as a subset of the space  $BV(\Omega)$ , the space of bounded variation functions, in order to recover discontinuous coefficients. On the other hand, the total variation of the control variables is assumed to be uniformly bounded, in order to ensure the compactness result useful to prove the existence of an optimal control. Then, we establish that the associated adjoint state system admits a unique solution. This retrograde problem consists in a strongly coupled system of linear elliptic–parabolic equations, the main difficulty is related to the presence of scalar products between the gradients of the exact solution and those of the unknowns. The  $L^4$ -regularity result obtained for the direct problem allows to deal with these terms. We thus use a Schauder fixed point theorem for proving the existence result. The second difficulty comes from the operator associating to the hydraulic conductivity the state variables. We have to establish that it is continuous and differentiable on suitable functional space. At this level, the main difficulty consists in defining the suitable function spaces so that the requirements of the implicit function theorem can be satisfied. Again, the  $L^\sigma$ -regularity of the exact solution (for  $\sigma > 4$ ) is fundamental in order to prove the continuity and differentiability of the previous operator. Finally, we establish the first order necessary optimality conditions.

This article is organized as follows: In section 2.1 we present the seawater intrusion problem and we recall the essential properties of the space of functions with bounded variations. We recall in section 3 all mathematical notations. Furthermore the global in time existence and uniqueness results are stated in confined case with sharp-diffuse interface approach. The section 4 contains the main results of the paper. We first prove the existence of the optimal control. Considering the system as a constraint for the optimization problem, we introduce the Lagrangian function associated with the cost function. We establish the existence and uniqueness results for the adjoint problem. We prove that the operator associating to the hydraulic conductivity  $K$  the solution,  $(f(K), h(K))$ , defined in section 2.1, is continuous and differentiable. The main point consists in finding the well adapted function spaces so that the implicit function theorem is applicable. Finally, we state the first order necessary optimality conditions for the optimization problem.

## 2 Problem statement and main results

### 2.1 Problem statement

We consider an open bounded domain  $\Omega$  of  $\mathbb{R}^2$  describing the projection of the aquifer on the horizontal plane. The boundary of  $\Omega$ , assumed  $\mathcal{C}^1$ , is denoted by  $\Gamma$ . The time interval of interest is  $(0, T)$ ,  $T$  being any nonnegative real number, and we set  $\Omega_\tau = (0, \tau) \times \Omega$ ,  $\forall \tau \in (0, T)$ . The confined aquifer is assumed to be bounded by two layers, the lower surface corresponds to  $z = h_2$  and the upper surface to  $z = h_1$ . Quantity  $h_2 - h_1$  represents the total thickness of the aquifer (Cf. Figure (1)). We assume that depths  $h_1, h_2$  are constant, such that  $h_2 > \delta_1 > 0$  and without loss of generality we can set  $h_1 = 0$ .

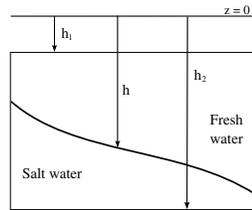


Figure 1 Schematization of an aquifer

In the case of confined aquifer, the well adapted unknowns are the interface depth  $h$  and the freshwater hydraulic head  $f$ . The nonnegative function  $K$  represents the hydraulic conductivity (which is assumed to be a scalar),  $\phi$  denotes the porosity,  $\delta$  the thickness of the diffuse saltwater/freshwater interface ( $\delta = 0$  corresponds to the classical sharp interface approach) and the parameter  $\alpha$  characterizes the densities contrast :

$$\alpha = \frac{\rho_s}{\rho_f} - 1,$$

where the characteristics  $\rho_s$  and  $\rho_f$  are respectively the densities of saltwater and freshwater. Thus, the model reads (see [10, 20]) :

$$\phi \partial_t h - \nabla \cdot (\alpha K (h_2 - h) \nabla h) - \nabla \cdot (\delta \nabla h) + \nabla \cdot (K (h_2 - h) \nabla f) = -Q_s, \quad (2.1)$$

$$-\nabla \cdot (h_2 K \nabla f) + \nabla \cdot (K (h_2 - h) \nabla h) = Q_f + Q_s. \quad (2.2)$$

In the previous system, the second equation models the conservation of total mass of water, while the first is modeling the mass conservation of saltwater. This is a 2D model, the third dimension being preserved by the upscaling process *via* the depth information  $h$ .

The system (2.1)- (2.2) is completed by the following boundary and initial conditions :

$$h = h_D, \quad f = f_D \quad \text{in } \Gamma \times (0, T), \quad (2.3)$$

$$h(0, x) = h_0(x) \quad \text{in } \Omega, \quad (2.4)$$

with the compatibility condition

$$h_0(x) = h_D(0, x), \quad x \in \Gamma.$$

The inverse problem is formulated by an optimization problem whose cost function measures the squared difference between experimental interfaces depth and those given by the model. We introduce the following control problem:

$$(\mathcal{O}) \begin{cases} \text{Find } K^* \in U_{adm} \text{ such that} \\ \mathcal{J}(K^*) = \inf_{K \in U_{adm}} \mathcal{J}(K), \end{cases}$$

where

$$\mathcal{J}(K) = \frac{1}{2} \|f(K) - f_{obs}\|_{L^2(\Omega_T)}^2 + \frac{1}{2} \|h(K) - h_{obs}\|_{L^2(\Omega_T)}^2,$$

where  $(f(K), h(K))$  denotes the solution of the variational problem (2.1)-(2.2) completed by the boundary and initial conditions (2.3)-(2.4). The functions  $(f_{obs}, h_{obs})$  correspond to the observed hydraulic head and depth of the freshwater/salt water interface.

We thus propose to work on the set of admissible parameters:

$$U_{adm} = \{K \in BV(\Omega) \cap L^\infty(\Omega), K_- \leq K \leq K_+ \text{ and } TV(K) \leq c\},$$

where  $K_-$ ,  $K_+$  and  $c$  are nonnegative real constants. This choice of set allows to recover discontinuous coefficients for the hydraulic conductivity.

We denote by  $BV(\Omega)$  the space of functions in  $L^1(\Omega)$  with bounded variation on  $\Omega$ , which is a Banach space for the norm

$$\|g\|_{BV(\Omega)} = \|g\|_1 + TV(g).$$

**Definition 1** Let  $g$  be a function of  $L^1(\Omega)$ ; we call total variation of  $g$  on  $\Omega$  the real

$$TV(g) = \sup \left\{ \int_{\Omega} g(x) \nabla \cdot v(x) dx, v \in C_0^\infty(\Omega)^n, \|v\|_\infty \leq 1 \right\}.$$

$g$  is a function with bounded variation if  $TV(g) < \infty$ .

We remind that  $U_{adm}$  is a compact subset of  $L^r(\Omega)$  for all  $r \in [1, +\infty[$ , we refer the reader to [11] for more details about that.

## 2.2 Main results

### Existence of optimal control

Since the total variation of the control variables is assumed to be uniformly delimited, this ensures a compactness result for the set  $U_{adm}$  in  $L^2(\Omega)$ . This property combined with the uniqueness of the exact solution results in the existence of a solution for the control problem.

**Theorem 2.1** There exists at least one optimal control for the problem  $(\mathcal{O})$ .

### Existence and uniqueness results for the adjoint problem

By considering the exact problem as a constraint for the optimization problem and introducing the Lagrangian associated with the cost function, we define the associated adjoint state problem given by the following retrograde system:

$$\begin{cases} -\phi \frac{\partial \lambda_i}{\partial t} - \nabla \cdot ((\delta + \alpha K T_s(h)) \nabla \lambda_i) - \alpha K(x) \nabla h \cdot \nabla \lambda_i + K(x) \nabla f \cdot \nabla \lambda_i \\ + \nabla \cdot (\alpha K(x) T_s(h) \nabla \lambda_f) + \alpha K(x) \nabla h \cdot \nabla \lambda_f = h_{obs} - h, \\ -\nabla \cdot (K(x) h_2 \nabla \lambda_f) + \nabla \cdot (K(x) T_s(h) \nabla \lambda_i) = f_{obs} - f, \end{cases} \quad (2.5)$$

$$\lambda_i = 0, \lambda_f = 0 \text{ on } \Gamma_D, \lambda_i(T, x) = 0, \forall x \in \Omega. \quad (2.6)$$

Then we prove the well-posedness of system (2.5)-(2.6):

**Theorem 2.2** Assuming that the hydraulic conductivity satisfies

$$K_+ \leq \frac{\delta}{\alpha h_2}. \quad (2.7)$$

Let  $(f, h)$  be the solution of (2.1)-(2.4) associated with the hydraulic conductivity  $K \in U_{adm}$ , the adjoint problem defined by

$$\begin{aligned} & \text{Find } (\lambda_i, \lambda_f) \in W(0, T) \times H_0^1(\Omega) \text{ such that } \lambda_i(T, \cdot) = 0 \text{ and } \forall (\varphi_f, \varphi_i) \in L^2(0, T; H_0^1(\Omega)^2) : \\ & \int_{\Omega_T} [-\phi \frac{\partial \lambda_i}{\partial t} \varphi_i + (\delta + \alpha K(x) T_s(h)) \nabla \lambda_i \cdot \nabla \varphi_i - \alpha K(x) T_s(h) \nabla \lambda_f \cdot \nabla \varphi_i] dx dt \\ & + \int_{\Omega_T} [K(x) (\nabla f - \alpha \nabla h) \cdot \nabla \lambda_i + \alpha K(x) \nabla h \cdot \nabla \lambda_f] \varphi_i dx dt = \int_{\Omega_T} (h_{obs} - h) \varphi_i dx dt, \end{aligned} \quad (2.8)$$

$$\int_{\Omega_T} [K(x) h_2 \nabla \lambda_f \cdot \nabla \varphi_f - K(x) T_s(h) \nabla \lambda_i \cdot \nabla \varphi_f] dx dt = \int_{\Omega_T} (f_{obs} - f) \varphi_f dx dt, \quad (2.9)$$

has a unique solution.

### Optimality conditions

In order to state the first order necessary optimality conditions for the problem  $(\mathcal{O})$ , we first aim to check that the operator, denoted by  $\mathcal{Q}$ , associating to the hydraulic conductivity  $K$  the solution  $(f(K), h(K))$  of (2.1)-(2.4) is continuous and differentiable on suitable function spaces. The main point consists in finding the well adapted function spaces so that the implicit function theorem is applicable. It can be possible if the gradient of the solution of the exact problem is sufficiently regular, namely if  $(\nabla h, \nabla f) \in (L^s(0, T; W^{1,s}(\Omega)))^2$  for  $s > 2$ . We thus introduce the following spaces

$$Z(0, T) = (L^2(0, T; H_0^1(\Omega)) \times W(0, T)) \cap (L^\infty(0, T; L^2(\Omega)))^2 \cap (L^s(0, T; W^{1,s}(\Omega)))^2, \quad (2.10)$$

and

$$U = \{K \in BV(\Omega) \cap L^\infty(\Omega), K_- \leq K \leq K_+ \text{ and } TV(K) \leq C\}, \text{ with } c < C, \quad (2.11)$$

where the constant  $c$  is the constant defining  $U_{adm}$  and  $s = 1 + \sqrt{5}$ . In these spaces, the following Theorem holds true

**Theorem 2.3** The mapping  $\mathcal{Q}$  is continuous and differentiable from  $U_{adm}$  to  $Z(0, T)$ .

By collecting all the previous results, we directly verify that the minimum required,  $K^*$ , satisfies the optimality system and thus establish the following result

**Theorem 2.4** Let  $K^*$  be a solution of problem  $(\mathcal{O})$ , there exists a couple  $(h^* - h_D, f^* - f_D) \in W(0, T) \times L^2(0, T, H_0^1(\Omega))$  and a couple  $\lambda^* = (\lambda_i^*, \lambda_f^*) \in W(0, T) \times L^2(0, T, H_0^1(\Omega))$  satisfying the optimality system determined by the direct problem (2.1)-(2.4), the adjoint problem (2.5)-(2.6) and, for all  $K \in U_{adm}$

$$D_K \mathcal{J}(K^*) \cdot (K(x) - K^*(x)) \geq 0, \quad (2.12)$$

with the gradient of the cost function given by,  $\forall \delta_K \in U_{adm}$

$$\begin{aligned} D_K \mathcal{J}(K^*) \cdot K &= \int_{\Omega_T} K [(h_2 - h^*)(\alpha \nabla h^* - \nabla f^*) \cdot \nabla \lambda_i^* \\ &\quad + (h_2 \nabla f^* - \alpha (h_2 - h^*) \nabla h^*) \cdot \nabla \lambda_f^*] dx dt. \end{aligned} \quad (2.13)$$

### 3 Global in time existence and uniqueness results

We introduce function  $T_s$  defined by

$$T_s(u) = h_2 - u \quad \forall u \in (\delta_1, h_2).$$

Function  $T_s$  is extended continuously and constantly outside  $(\delta_1, h_2)$ . It represents the thickness of the saltwater zone in the reservoir, the previous extension of  $T_s$  for  $h \leq \delta_1$  enables to ensure a thickness of freshwater zone always greater than  $\delta_1$  inside the aquifer. Let us now detail the mathematical assumptions. We begin with the characteristics of the porous structure. We assume the existence of two positive real numbers  $K_-$  and  $K_+$  such that the hydraulic conductivity  $K$  is nonnegative and uniformly bounded, namely

$$0 < K_- \leq K(x) \leq K_+ < \infty \quad x \in \Omega.$$

We suppose that porosity  $\phi$  is constant in the aquifer. Indeed, in the field envisaged here, the effects due to variations in  $\phi$  are negligible compared with those due to density contrasts. From a mathematical point of view, these assumptions do not change the complexity of the analysis but rather avoid cumbersome computations. We remind that the parameter  $\delta$  represents the thickness of the diffuse saltwater/freshwater interface, it is of order of 1 m.

Source terms  $Q_f$  and  $Q_s$  are given functions of  $L^2(0, T; H)$  and we assume that  $Q_f \geq 0$  and  $Q_s \leq 0$ .

Functions  $h_D$  and  $f_D$  belong to the space  $(L^2(0, T; H^1(\Omega)) \cap H^1(0, T; (H^1(\Omega))')) \times L^2(0, T; H^1(\Omega))$  while the function  $h_0$  is in  $H^1(\Omega)$ . Finally, we assume that the boundary and initial data satisfy conditions on the hierarchy of interfaces depths:

$$0 < \delta_1 \leq h_D \leq h_2 \text{ a.e. in } \Gamma \times (0, T), \quad 0 < \delta_1 \leq h_0 \leq h_2 \text{ a.e. in } \Omega.$$

We first recall the global existence result established in [10] :

**Theorem 3.1** Assume a low spatial heterogeneity for the hydraulic conductivity tensor:

$$K_+ < \frac{h_2}{h_2 - \delta_1} \inf \left( \sqrt{\frac{\delta K_-}{3 h_2}}, K_- \right). \quad (3.1)$$

Then for any  $T > 0$ , problem (2.1)-(2.4) admits a weak solution  $(h, f)$  satisfying

$$(h - h_D, f - f_D) \in W(0, T) \times L^2(0, T; H_0^1(\Omega)).$$

Furthermore the following maximum principle holds true:

$$0 < \delta_1 \leq h(t, x) \leq h_2 \quad \text{for a.e. } x \in \Omega \text{ and for any } t \in (0, T).$$

**Remark 1** We emphasize that the depth  $h$  is naturally bounded by two quantities characterizing the aquifer as shown by the maximum principle established in Theorem 3.1. This result is specific to confined aquifers, it is no longer valid in the case of free aquifers, for which one can find oneself in situations of overflow of the aquifer.

The following uniqueness result is a consequence of a  $L^r(0, T; W^{1,r}(\Omega))$ ,  $r > 2$ , regularity result proved for the solution of (2.1)-(2.2). This regularity is a generalization of the Meyers regularity results [16] given in elliptic case and extended in parabolic case in [3]. We first introduce some useful notations.

• **Elliptic case**

We recall the following result (see J.L. Lions and E. Magenes [14]):

$$\forall p \text{ s.t. } 1 < p < \infty, -\Delta \text{ is an isomorphism from } W_0^{1,p}(\Omega) \text{ to } W^{-1,p}(\Omega).$$

We set  $G = (-\Delta)^{-1}$  and  $g(p) = \|G\|_{\mathcal{L}(W^{-1,p}(\Omega); W_0^{1,p}(\Omega))}$ . We notice that  $g(2)=1$ . We thus introduce, for any real number  $c > 0$

$$\mu = \frac{K_- + c}{K_+ + c} \text{ and } \nu = \frac{c}{(K_+ + c)}, \quad (3.2)$$

the positivity of  $c$  ensuring  $\nu < \mu$ . We consider  $r, r > 2$  such that  $k(r) := g(r)(1 - \mu + \nu) < 1$ . Letting  $c \rightarrow 0$ , the condition  $k(r) < 1$  yields

$$K_+ < \frac{g(r)}{g(r) - 1} K_-. \quad (3.3)$$

The condition (3.3) implies a low spatial heterogeneity for the hydraulic conductivity, so as the assumption (3.1).

• **Parabolic case**

We set  $P = \frac{\partial}{\partial t} - \Delta$ , the operator associated with the homogeneous Dirichlet boundary conditions. We know that, being given  $F \in Y_p$ , there is a unique solution  $u \in X_p$  such that:

$$Pu = F \text{ in } \Omega_T, \quad u(0) = u_0.$$

We denote  $\hat{g}(p) = \|P^{-1}\|_{\mathcal{L}(Y_p; X_p)}$  and we recall that  $\hat{g}(2) = 1$ .

Again we introduce, for any real number  $\hat{c} > 0$

$$\hat{\mu} = \frac{\delta + \phi \hat{c}}{\delta + \phi \hat{c} + K_+(h_2 - \delta_1)} \text{ and } \hat{\nu} = \frac{\phi \hat{c}}{\delta + \phi \hat{c} + K_+(h_2 - \delta_1)}. \quad (3.4)$$

Since  $\hat{c} > 0$ , we get  $\hat{\nu} < \hat{\mu}$ . We consider the exponent  $r > 2$  such that  $\hat{k}(r) := \hat{g}(r)(1 - \hat{\mu} + \hat{\nu}) < 1$ . Letting  $\hat{c} \rightarrow 0$ , the assumption  $\hat{k}(r) < 1$  leads to

$$\frac{K_+(h_2 - \delta_1)}{\delta} < \frac{1}{\hat{g}(r) - 1}. \quad (3.5)$$

We thus can state the following Theorem (cf [10]):

**Theorem 3.2** Let  $(h_2, K_-, K_+, \delta, \delta_1) \in (\mathbb{R}_*^+)^5$  satisfying (3.1), (3.3) and (3.5) for  $r = 4$ . Furthermore we assume that there exists  $\gamma, 0 < \gamma < 1$ , such that the physical parameters satisfy

$$\frac{K_+(h_2 - \delta_1)}{\delta} \leq (1 - \gamma) \times \frac{(1 - k(4))}{g(4)} \times \frac{(1 - \hat{k}(4))}{\hat{g}(4)} \times \frac{h_2}{h_2 - \delta_1}. \quad (3.6)$$

If  $h_0 \in W^{1,4}(\Omega)$ ,  $(h_D, f_D) \in L^4(0, T; W^{1,4}(\Omega))^2$  and  $(Q_s, Q_f) \in L^4(\Omega_T)^2$ , then the solution of the system (2.1)-(2.4) is unique in  $(W(0, T) + h_D) \times (L^2(0, T; H^1(\Omega)) + f_D)$ . Moreover, we have

$$\|\nabla h\|_{(L^4(\Omega_T))^2} \leq C_4(\phi, h_2, h_0, h_D, f_D, Q_s, Q_f, K_-, K_+, \delta, \delta_1), \quad (3.7)$$

$$\|\nabla f\|_{(L^4(\Omega_T))^2} \leq C_4(\phi, h_2, h_0, h_D, f_D, Q_s, Q_f, K_-, K_+, \delta, \delta_1). \quad (3.8)$$

**Remark 2** Assumption (3.1) (so as (3.3) and (3.5)) makes only sense when considering low values for  $K$ . For the present application, this point is not restrictive since the soil permeability typically ranges from  $10^{-8}$  to  $10^{-3} \text{ m} \cdot \text{s}^{-1}$ . A priori, assumption (3.6) is stronger than (3.5) except if  $\delta_1$ , the thickness of freshwater zone inside the aquifer, is sufficiently large.

By now, we suppose that the hydraulic conductivity  $K$  satisfies (3.1), (3.3), (3.5) and (3.6) in order to guarantee the existence and the uniqueness of the solution of (2.1)-(2.4).

## 4 Identification of hydraulic conductivity

In this section, we prove the results that constitute the four steps to solve this inverse problem. We first of all show that this exact problem admits a solution. Then, by considering the exact problem as a constraint for the optimization problem and introducing the Lagrangian associated with the cost function, the solution is characterized by the optimality system it must satisfy.

### 4.1 Proof of Theorem 2.1

The two keys to this proof are on the one hand the compactness of  $U_{adm}$  in  $L^2(\Omega)$  and on the other hand the unicity of the solution of the exact problem.

Let  $(K_n)_{n \in \mathbb{N}} \subset U_{adm}$  be a minimizing sequence such that

$$\mathcal{J}(K_n) \longrightarrow \mathcal{J}^* = \inf_{K \in U_{adm}} \mathcal{J}(K).$$

Since  $U_{adm}$  is a compact subset of  $L^2(\Omega)$ , we deduce there exists a subsequence, still denoted  $K_n$ , and a function  $K^* \in U_{adm}$  such that

$$K_n \longrightarrow K^* \text{ strongly in } L^2(\Omega). \quad (4.1)$$

In an other hand, the solution  $(f^n, h^n) = (f(K_n), h(K_n))$  of the variational problem, satisfies:

$$\|f^n\|_{L^2(0, T, H^1(\Omega))} + \|h^n\|_{L^2(0, T, H^1(\Omega))} \leq C, \quad \delta_1 \leq h^n \leq h_2, \quad (4.2)$$

$$\|\partial_t h^n\|_{L^2(0, T, V')} \leq C, \quad (4.3)$$

where  $C$  is a constant independent on  $n$ .

Then  $(h^n)_n$  is uniformly bounded in  $W(0, T)$ , we deduce from Aubin compactness result that the sequence  $(h^n - h_D)_n$  is sequentially compact in  $L^2(0, T, H)$ . Furthermore  $(f^n)_n$  is sequentially weakly compact in  $L^2(0, T; H^1(\Omega))$ .

We can therefore extract a subsequence, not relabeled for convenience,  $(f^n, h^n - h_D)_n \in$

$L^2(0, T; H^1(\Omega)) \times W(0, T)$  and there exists  $(f^*, h^* - h_D) \in L^2(0, T; H^1(\Omega)) \times W(0, T)$  such that :

$$\begin{aligned} h^n &\longrightarrow h^* \text{ in } L^2(0, T; H) \text{ and a.e. in } [0, T] \times \Omega, \\ \partial_t h^n &\longrightarrow \partial_t h^* \text{ weakly in } L^2(0, T; V'), \\ f^n &\longrightarrow f^* \text{ in } L^2(0, T; H^1(\Omega)). \end{aligned}$$

This allows to pass to the limit in the variational formulation corresponding to exact problem (2.1)-(2.4).

From the uniqueness of the solution, it follows that:

$$(f^*, h^*) = (f(K^*), h(K^*)) \text{ and then } \mathcal{J}(K^*) = \mathcal{J}^*.$$

This ends the proof.

## 4.2 Definition of the Lagrangian $\mathcal{L}$

We must now determine the minimum of the cost function  $\mathcal{J}$ , the state system being the unsteady problem (2.1)-(2.4). We consider this system as a constraint for the optimization problem, where the Lagrangian  $\mathcal{L}$  is defined as follows :

$$\begin{aligned} \mathcal{L}(K, f, h, \lambda_f, \lambda_i) &= \mathcal{J}(K) + \int_0^T \int_{\Omega} \phi \frac{\partial h}{\partial t} \lambda_i \, dx dt \\ &+ \int_0^T \int_{\Omega} (\delta + \alpha K(x) T_s(h)) \nabla h \cdot \nabla \lambda_i \, dx dt - \int_0^T \int_{\Omega} K(x) T_s(h) \nabla f \cdot \nabla \lambda_i \, dx dt \\ &+ \int_0^T \int_{\Omega} K(x) h_2 \nabla f \cdot \nabla \lambda_f \, dx dt - \int_0^T \int_{\Omega} \alpha K(x) T_s(h) \nabla h \cdot \nabla \lambda_f \, dx dt \\ &+ \int_0^T \int_{\Omega} Q_s \lambda_i \, dx dt - \int_0^T \int_{\Omega} (Q_s + Q_f) \lambda_f \, dx dt. \end{aligned} \tag{4.4}$$

The solution thus corresponds to a saddle point of  $\mathcal{L}$  considered as function of independent variables  $h, f, \lambda_i, \lambda_f$  and  $K$  with  $\lambda_i$  and  $\lambda_f$  the Lagrange multipliers. The minimum,  $K^*$ , satisfies the following optimality system :

$$\left\{ \begin{array}{ll} \frac{\partial \mathcal{L}}{\partial \lambda_i}(K^*, f^*, h^*, \lambda_f^*, \lambda_i^*) = 0, & \frac{\partial \mathcal{L}}{\partial \lambda_f}(K^*, f^*, h^*, \lambda_f^*, \lambda_i^*) = 0, \\ \frac{\partial \mathcal{L}}{\partial h}(K^*, f^*, h^*, \lambda_f^*, \lambda_i^*) = 0, & \frac{\partial \mathcal{L}}{\partial f}(K^*, f^*, h^*, \lambda_f^*, \lambda_i^*) = 0, \\ \frac{\partial \mathcal{L}}{\partial K}(K^*, f^*, h^*, \lambda_f^*, \lambda_i^*) \cdot (K - K^*) \geq 0, & \forall K \in U_{adm}. \end{array} \right. \tag{4.5}$$

where  $(h^*, f^*)$  is the unique solution of (2.1)-(2.4) for  $K = K^*$  and  $(\lambda_f^*, \lambda_i^*)$  the unique solution of the adjoint system (2.5)-(2.6) for  $K = K^*$ .

## 4.3 Adjoint problem - Proof of Theorem 2.2

Since the system (2.5)-(2.6) is retrograde, we firstly set  $t' = T - t$ , the system thus becomes:

$$\int_{\Omega_T} \left[ \phi \frac{\partial \lambda_i}{\partial t'} \varphi_i + (\delta + \alpha K(x) T_s(h)) \nabla \lambda_i \cdot \nabla \varphi_i - \alpha K(x) T_s(h) \nabla \lambda_f \cdot \nabla \varphi_i \right] dx dt$$

$$+ \int_{\Omega_T} [K(x)(\nabla f - \alpha \nabla h) \cdot \nabla \lambda_i + \alpha K(x) \nabla h \cdot \nabla \lambda_f] \varphi_i dxdt = \int_{\Omega_T} (h_{obs} - h) \varphi_i dxdt, \quad (4.6)$$

and

$$\int_{\Omega_T} [K(x) h_2 \nabla \lambda_f \cdot \nabla \varphi_f - K(x) T_s(h) \nabla \lambda_i \cdot \nabla \varphi_f] dxdt = \int_{\Omega_T} (f_{obs} - f) \varphi_f dxdt. \quad (4.7)$$

The initial condition is now written as  $\lambda_i(0, x) = 0, \forall x \in \Omega$ .

This is a coupled system of elliptic-parabolic linear equations. From now on, we omit the prime in  $t'$ .

To prove the existence of the solution of (4.6)-(4.7), we will proceed as for the proof of global existence in confined case with diffuse interface (cf. [10]). We emphasize that the only difficulty, in this case, is the presence of linear terms  $K(\nabla f - \alpha \nabla h) \cdot \nabla \lambda_i$  and  $K \nabla h \cdot \nabla \lambda_f$  which simultaneously involve the gradients of the solutions of the exact problem and the adjoint problem. The result of regularity (3.7)-(3.8) established in Theorem 3.2 makes it possible to overcome this technical difficulty. In addition, since the system (2.5)-(2.6) is strongly coupled, a fixed point strategy is used to reduce it to the classic theory of linear problems.

### Global in time existence

For the strategy of fixed point, we introduce the application  $\mathcal{F}$  :

$$\begin{aligned} \mathcal{F} : L^2(0, T; H_0^1(\Omega))^2 &\longrightarrow L^2(0, T; H_0^1(\Omega))^2, \\ (\bar{\lambda}_i, \bar{\lambda}_f) &\rightarrow \mathcal{F}(\bar{\lambda}_i, \bar{\lambda}_f) = (\lambda_i, \lambda_f), \end{aligned}$$

where  $\lambda_i = \mathcal{F}_1(\bar{\lambda}_i, \bar{\lambda}_f)$  and  $\lambda_f = \mathcal{F}_2(\bar{\lambda}_i, \bar{\lambda}_f)$  are solutions of the variational problem,  $\forall w \in L^2(0, T; H_0^1(\Omega))$  :

$$\begin{aligned} &\int_0^T \phi \langle \partial_t h, \lambda_i \rangle_{V', V} dt + \int_{\Omega_T} (\delta + \alpha K T_s(h)) \nabla \lambda_i \cdot \nabla w dxdt \\ &- \alpha \int_{\Omega_T} K(x) T_s(h) \nabla \bar{\lambda}_f \cdot \nabla w dxdt + \int_{\Omega_T} K(x) (\nabla f - \alpha \nabla h) \cdot \nabla \bar{\lambda}_i w dxdt \\ &+ \alpha \int_{\Omega_T} K(x) \nabla \bar{\lambda}_f \cdot \nabla h w dxdt = - \int_{\Omega_T} (h - h_{obs}) w dxdt \end{aligned} \quad (4.8)$$

and

$$\int_{\Omega_T} K(x) h_2 \nabla \lambda_f \cdot \nabla w dxdt - \int_{\Omega_T} K(x) T_s(h) \nabla \lambda_i \cdot \nabla w dxdt = - \int_{\Omega_T} (f - f_{obs}) w dxdt \quad (4.9)$$

We know from the classical theory of linear parabolic equations that the previous variational system has a unique solution. We first have to prove the continuity of  $\mathcal{F}$ , that is the continuities of  $\mathcal{F}_1$  and  $\mathcal{F}_2$ .

**Sequential continuity of  $\mathcal{F}_1$  in  $L^2(0, T; H)$  when  $\mathcal{F}$  is restricted to any bounded subset of  $W(0, T) \times L^2(0, T; H^1(\Omega))$ .**

Let  $(\bar{\lambda}_i^n, \bar{\lambda}_f^n)$  be a bounded sequence of  $L^2(0, T; H_0^1(\Omega))^2$  and  $(\bar{\lambda}_i, \bar{\lambda}_f) \in L^2(0, T; H_0^1(\Omega))^2$  such that  $(\bar{\lambda}_i^n, \bar{\lambda}_f^n) \rightarrow (\bar{\lambda}_i, \bar{\lambda}_f)$  in  $L^2(0, T; L^2(\Omega))$ .

We thus have

$$(\bar{\lambda}_i^n, \bar{\lambda}_f^n) \rightharpoonup (\bar{\lambda}_i, \bar{\lambda}_f) \text{ weakly in } W(0, T) \times L^2(0, T; H^1(\Omega));$$

that is,  $\bar{\lambda}_i^n \rightharpoonup \bar{\lambda}_i$  weakly in  $L^2(0, T, V)$  (the same for  $\bar{\lambda}_f^n$  and  $\bar{\lambda}_f$ ) and  $\partial_t \bar{\lambda}_i^n \rightharpoonup \partial_t \bar{\lambda}_i$  weakly in  $L^2(0, T, V')$ . Set  $\lambda_{i,n} = \mathcal{F}_1(\bar{\lambda}_i^n, \bar{\lambda}_f^n)$  and  $\lambda_i = \mathcal{F}_1(\bar{\lambda}_i, \bar{\lambda}_f)$ . We first intend to show that  $\lambda_{i,n} \rightharpoonup \lambda_i$  weakly in  $W(0, T)$  and thus strongly in  $L^2(0, T; H)$  thanks to a classical result of Aubin. Pick a constant  $M > 0$ , that we will precise later on, such that

$$\|\nabla \bar{\lambda}_i^n\|_{(L^2(0,T;H))^2} \leq M \quad \text{and} \quad \|\nabla \bar{\lambda}_f^n\|_{(L^2(0,T;H))^2} \leq M. \quad (4.10)$$

For all  $n \in \mathbb{N}$ ,  $\lambda_{i,n}$  satisfies (4.8). Pick any  $T > 0$  and take  $w = \lambda_{i,n} \chi_{(0,T)}(t)$  in (4.8). We obtain

$$\begin{aligned} & \frac{\phi}{2} (\|\lambda_{i,n}\|_H^2 - \underbrace{\|\lambda_{i,n}(0, \cdot)\|_H^2}_{=0}) + \underbrace{\int_{\Omega_T} (\delta + \alpha K T_s(h)) \nabla \lambda_{i,n} \cdot \nabla \lambda_{i,n} \, dx dt}_{(1)} \\ & - \alpha \underbrace{\int_{\Omega_T} K(x) T_s(h) \nabla \bar{\lambda}_f^n \cdot \nabla \lambda_{i,n} \, dx dt}_{(2)} + \underbrace{\int_{\Omega_T} K(x) (\nabla f - \alpha \nabla h) \cdot \nabla \bar{\lambda}_i^n \lambda_{i,n} \, dx dt}_{(3)} \\ & + \alpha \underbrace{\int_{\Omega_T} K(x) \nabla \bar{\lambda}_f^n \cdot \nabla h \lambda_{i,n} \, dx dt}_{(4)} = - \underbrace{\int_{\Omega_T} (h - h_{obs}) \lambda_{i,n} \, dx dt}_{(5)}. \end{aligned}$$

Clearly (1)  $\geq \delta \|\nabla \lambda_{i,n}\|_{L^2(\Omega_T)}^2$ .

Moreover, thanks to Cauchy-Schwarz, Young and Gagliardo-Nirenberg inequalities, we get  $\forall \epsilon > 0$

$$\begin{aligned} |(2)| & \leq \alpha K_+ h_2 M \sqrt{T} \|\nabla \lambda_{i,n}\|_{L^2(\Omega_T)} \leq \frac{\alpha^2 K_+^2 h_2^2 M^2 T}{2\epsilon} + \frac{\epsilon}{2} \|\nabla \lambda_{i,n}\|_{L^2(\Omega_T)}^2, \\ |(3)| & \leq \int_0^T M \left| \int_{\Omega} K(x) (\nabla f - \alpha \nabla h) \cdot \nabla \bar{\lambda}_i^n \lambda_{i,n} \, dx dt \right| \\ & \leq \int_0^T M K_+ \left\{ \left( \int_{\Omega} |\nabla f|^4 \, dx \right) + \alpha \left( \int_{\Omega} |\nabla h|^4 \, dx \right) \right\}^{1/4} \left( \int_{\Omega} \lambda_{i,n}^4 \, dx \right)^{1/4} \left( \int_{\Omega} |\nabla \bar{\lambda}_i^n|^2 \right)^{1/2} dt \\ & \leq C_G \int_0^T K_+ M \left( \int_{\Omega} |\nabla f|^4 \, dx + \alpha \int_{\Omega} |\nabla h|^4 \, dx \right)^{1/4} \left( \int_{\Omega} |\nabla \lambda_{i,n}|^2 \, dx \right)^{1/4} \left( \int_{\Omega} \lambda_{i,n}^2 \, dx \right)^{1/4} \left( \int_{\Omega} |\nabla \bar{\lambda}_i^n|^2 \right)^{1/2} dt \\ & \leq M \|\nabla \bar{\lambda}_i^n\|_{L^2(\Omega_T)} K_+ C_G \underbrace{\left( \|\nabla f\|_{L^4(\Omega_T)} + \alpha \|\nabla h\|_{L^4(\Omega_T)} \right)}_{\leq C_4(1+\alpha)} \left( \int_0^T \|\lambda_{i,n}\|_{L^2(\Omega)}^2 \|\nabla \lambda_{i,n}\|_{L^2(\Omega)}^2 dt \right)^{1/4} \\ & \leq K_+^2 C_4^2 (1+\alpha)^2 C_G^2 M^2 \times \frac{1}{2\epsilon} + \frac{\epsilon}{2} \left( \int_0^T \|\lambda_{i,n}\|_{L^2(\Omega)}^2 \|\nabla \lambda_{i,n}\|_{L^2(\Omega)}^2 dt \right)^{1/2} \\ & \leq \frac{K_+^2 C_4^2 (1+\alpha)^2 C_G^2 M^2}{2\epsilon} + \frac{\epsilon}{4} \max_{t \in (0,T)} \|\lambda_{i,n}\|_{L^2(\Omega)}^2 + \frac{\epsilon}{4} \int_{\Omega_T} |\nabla \lambda_{i,n}|^2. \end{aligned}$$

In the same way :

$$\begin{aligned} |(4)| & \leq \alpha K_+ M \int_0^T \left( \int_{\Omega} |\nabla h|^4 \, dx \right)^{1/4} \left( \int_{\Omega} \lambda_{i,n}^4 \, dx \right)^{1/4} \left( \int_{\Omega} |\nabla \bar{\lambda}_f^n|^2 \right)^{1/2} dt \\ & \leq \alpha K_+ M C_4 C_G \left( \int_0^T \|\lambda_{i,n}\|_{L^2(\Omega)}^2 \|\nabla \lambda_{i,n}\|_{L^2(\Omega)}^2 dt \right)^{1/4} \\ & \leq \alpha^2 K_+^2 M^2 C_4^2 C_G^2 \times \frac{1}{2\epsilon} + \frac{\epsilon}{2} \left( \int_0^T \|\lambda_{i,n}\|_{L^2(\Omega)}^2 \|\nabla \lambda_{i,n}\|_{L^2(\Omega)}^2 dt \right)^{1/2} \\ & \leq \frac{\alpha^2 K_+^2 C_4^2 C_G^2 M^2}{2\epsilon} + \frac{\epsilon}{4} \max_{t \in (0,T)} \|\lambda_{i,n}\|_{L^2(\Omega)}^2 + \frac{\epsilon}{4} \|\nabla \lambda_{i,n}\|_{L^2(\Omega_T)}^2, \end{aligned}$$

finally

$$|(5)| \leq \frac{1}{2\epsilon} \int_{\Omega_T} (h - h_{obs})^2 dx + \frac{\epsilon}{2} \max_{t \in (0, T)} \|\lambda_{i,n}\|_{L^2(\Omega)}^2.$$

By gathering together all these inequalities, we obtain

$$\frac{\phi}{2} \|\lambda_{i,n}\|_{L^2(\Omega)}^2 + (\delta - \epsilon) \|\nabla \lambda_{i,n}\|_{2L^2(\Omega_T)}^2 \leq \epsilon \max_{t \in (0, T)} \|\lambda_{i,n}\|_{L^2(\Omega)}^2 + \tilde{C}_M$$

where  $\tilde{C}_M$  is a constant depending on the data and on  $M$ .

Choosing  $\epsilon > 0$  such that  $\delta - \epsilon > 0$  and  $\frac{\phi}{2} - \epsilon > 0$ , we established that there are two reals  $A_M(h, h_{obs}, f, K, \alpha, C_4, \delta, \phi, h_2, T)$  and  $B_M(h, h_{obs}, f, K, \alpha, C_4, \delta, \phi, h_2, T)$  depending only on the data such that

$$\|\lambda_{i,n}\|_{L^\infty(0, T; H)}^2 \leq A_M \text{ and } \|\lambda_{i,n}\|_{L^2(0, T; H_0^1(\Omega))}^2 \leq B_M. \quad (4.11)$$

The sequence  $(\lambda_{i,n})_n$  thus is uniformly bounded in  $L^\infty(0, T; H) \cap L^2(0, T; H_0^1(\Omega))$ . We set  $C_M = \max(A_M, B_M)$ .

We now establish that  $(\partial_t \lambda_{i,n})_n$  is bounded in  $L^2(0, T; V') \cap L^1(0, T; H)$ , so we have

$$\begin{aligned} \phi \|\partial_t \lambda_{i,n}\|_{L^2(0, T; V') \cap L^1(0, T; H)} = & \\ \sup_{\|w\|_{(L^2(0, T, V) \cap L^\infty(0, T, H))} \leq 1} & \underbrace{\left| - \int_{\Omega_T} (\delta + \alpha K T_s(h)) \nabla \lambda_{i,n} \cdot \nabla w dx dt \right|}_{(1)} \\ + \alpha \underbrace{\int_{\Omega_T} K(x) T_s(h) \nabla \bar{\lambda}_f^n \cdot \nabla w}_{(2)} & - \underbrace{\int_{\Omega_T} K(x) (\nabla f - \alpha \nabla h) \cdot \nabla \bar{\lambda}_i^n w}_{(3)} dx dt \\ - \alpha \underbrace{\int_{\Omega_T} K(x) \nabla \bar{\lambda}_f^n \cdot \nabla h w}_{4} dx dt & - \underbrace{\int_{\Omega_T} (h - h_{obs}) w}_{(5)} dx dt \end{aligned}$$

As the principle of the calculations is the same as for the above inequalities, we only provide key estimates:

$$\begin{aligned} |(1)| & \leq (\delta + \alpha K_+ h_2) C_M \|w\|_{L^2(0, T; V)}, \\ |(2)| & \leq \alpha K_+ h_2 M \|w\|_{L^2(0, T; V)}, \\ |(3)| & \leq (1 + \alpha) K_+ M C_4 C_G \|w\|_{L^\infty(0, T; H)}^{1/4} \|w\|_{L^2(0, T; V)}^{1/2}, \\ |(4)| & \leq \alpha K_+ M C_4 C_G \|w\|_{L^\infty(0, T; H)}^{1/4} \|w\|_{L^2(0, T; V)}^{1/2}, \\ |(5)| & \leq \|h_{obs} - h\|_{L^2(\Omega_T)} \|w\|_{L^2(0, T; V)}. \end{aligned}$$

We can conclude that

$$\|\partial_t \lambda_{i,n}\|_{L^2(0, T; V) \cap L^1(0, T; H)} \leq D_M,$$

where  $D_M$  depends only on the data and on  $M$ .

Thus  $(\lambda_{i,n})_n$  is uniformly bounded in the space  $L^2(0, T; H^1(\Omega)) \cap H^1(0, T; V')$ .

Using the Aubin Lemma, we can extract a sequence  $(\lambda_{i,n})_n$ , not relabeled for simplicity, that strongly converges in  $L^2(\Omega_T)$  and weakly in  $L^2(0, T; H_0^1(\Omega)) \cap H^1(0, T; V')$  to a limit  $\lambda_l$ .

Thanks to the strong convergence of  $(\lambda_{i,n})_n$  in  $L^2(\Omega_T)$  (and then the convergence a.e. in  $\Omega_T$ ), we can check that  $\lambda_l$  is a solution of (4.8). The solution of (4.8) being unique, we have  $\lambda_i = \lambda_l$

and that the whole sequence  $\lambda_{i,n} \rightarrow \lambda_i$  weakly in  $W(0, T)$  and strongly in  $L^2(0, T; H)$ . The sequential continuity of  $\mathcal{F}_1$  in  $L^2(0, T; H)$  is established.

**Continuity of  $\mathcal{F}_2$  in  $L^2(0, T; H)$  when  $\mathcal{F}$  is restricted to any bounded subset of  $W(0, T) \times L^2(0, T; H^1(\Omega))$ :**

Similarly,, we study the sequential continuity of  $\mathcal{F}_2$  by setting  $\lambda_{f,n} := \mathcal{F}_2(\bar{\lambda}_i^n, \bar{\lambda}_f^n)$ ,  $\lambda_f := \mathcal{F}_2(\bar{\lambda}_i, \bar{\lambda}_f)$ , and showing first that  $\lambda_{f,n} \rightarrow \lambda_f$  in  $L^2(0, T; H^1(\Omega))$  weakly. Key estimates are obtained using the same type of arguments as those used to prove the sequential continuity of  $\mathcal{F}_1$ . The details are therefore omitted. We only point out that we can use the estimate (4.17) previously derived for  $\lambda_{i,n}$  to obtain the following estimates for  $\lambda_{f,n}$ , for  $\epsilon > 0$ :

$$h_2 \int_{\Omega_T} |\nabla \lambda_{f,n}|^2 \leq \epsilon \int_{\Omega_T} |\nabla \lambda_{f,n}|^2 + K_+^2 h_2^2 \int_{\Omega_T} |\nabla \lambda_{i,n}|^2 + C_p^2 \int_{\Omega_T} (f - f_{obs})^2,$$

where  $C_p$  is the constant in Poincaré’s inequality.

Choosing  $\epsilon$  such that  $K_- h_2 - \epsilon > 0$ , we get

$$\int_{\Omega_T} |\nabla \lambda_{f,n}|^2 \leq \underbrace{\frac{1}{K_- h_2 - \epsilon} (K_+^2 h_2^2 B_M + C_p^2 \|f - f_{obs}\|_{L^2(\Omega_T)}^2)}_{:=E_M}. \tag{4.12}$$

For proving the sequential compactness of  $\lambda_{f,n}$  in  $L^2(0, T; H)$ , we need some further work since we can not use a Aubin’s compactness *criterium* in the elliptic context characterizing  $\lambda_{f,n}$ . We actually get a stronger result: we claim and prove that  $\lambda_{f,n}$  converges in  $L^2(0, T; H^1(\Omega))$ . Indeed, we recall that the variational formulations defining respectively  $\lambda_{f,n}$  and  $\lambda_f$  are, for any  $w \in L^2(0, T; V)$ ,

$$\int_{\Omega_T} h_2 K \nabla \lambda_{f,n} \cdot \nabla w \, dxdt - \int_{\Omega_T} K T_s(h) \nabla \lambda_{i,n} \cdot \nabla w \, dxdt - \int_{\Omega_T} (h - h_{obs}) w \, dxdt = 0, \tag{4.13}$$

$$\int_{\Omega_T} h_2 K \nabla \lambda_f \cdot \nabla w \, dxdt - \int_{\Omega_T} K T_s(h) \nabla \lambda_i \cdot \nabla w \, dxdt - \int_{\Omega_T} (f - f_{obs}) w \, dxdt = 0. \tag{4.14}$$

The system (4.13)-(4.14) can be written as follows

$$\int_{\Omega_T} h_2 K \nabla \lambda_{f,n} \cdot \nabla w \, dxdt - \int_{\Omega_T} K \nabla (T_s(h) \lambda_{i,n}) \cdot \nabla w \, dxdt \tag{4.15}$$

$$- \int_{\Omega_T} K \lambda_{i,n} \nabla h \cdot \nabla w \, dxdt - \int_{\Omega_T} (h - h_{obs}) w \, dxdt = 0$$

$$\int_{\Omega_T} h_2 K \nabla \lambda_f \cdot \nabla w \, dxdt - \int_{\Omega_T} K \nabla (T_s(h) \lambda_i) \cdot \nabla w \, dxdt \tag{4.16}$$

$$- \int_{\Omega_T} K \lambda_i \nabla h \cdot \nabla w \, dxdt - \int_{\Omega_T} (f - f_{obs}) w \, dxdt = 0.$$

Choosing  $w = h_2 \lambda_{f,n} - T_s(h) \lambda_{i,n}$  in (4.15), we let  $n \rightarrow \infty$ . The already known convergence results let us pass to the limit in

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{\Omega_T} K \lambda_{i,n} \nabla h \cdot \nabla (h_2 \lambda_{f,n} - T_s(h) \lambda_{i,n}) \, dxdt - \int_{\Omega_T} (f - f_{obs}) (h_2 \lambda_{f,n} - T_s(h) \lambda_{i,n}) \, dxdt \\ & = \int_{\Omega_T} K \lambda_i \nabla h \cdot \nabla (h_2 \lambda_f - T_s(h) \lambda_i) \, dxdt - \int_{\Omega_T} (f - f_{obs}) (h_2 \lambda_f - T_s(h) \lambda_i) \, dxdt \end{aligned}$$

Moreover using (4.16) for the test function  $w = h_2 \lambda_f - T_s(h) \lambda_i$ , we conclude that

$$\lim_{n \rightarrow \infty} \int_{\Omega_T} K \nabla (h_2 \lambda_{f,n} - T_s(h) \lambda_{i,n}) \cdot \nabla (h_2 \lambda_{f,n} - T_s(h) \lambda_{i,n}) \, dxdt$$

$$= \int_{\Omega_T} K \nabla (h_2 \lambda_f - T_s(h) \lambda_i) \cdot \nabla (h_2 \lambda_f - T_s(h) \lambda_i) dx dt.$$

It follows that

$$\lim_{n \rightarrow \infty} \int_{\Omega_T} K \nabla (F_n - F) \cdot \nabla (F_n - F) dx dt = 0,$$

if  $F_n = h_2 \lambda_{f,n} - T_s(h) \lambda_{i,n}$  and  $F = h_2 \lambda_f - T_s(h) \lambda_i$ . Since  $K \xi \cdot \xi \geq K_- |\xi|^2$  for any  $\xi \in \mathbb{R}^2$  with  $K_- > 0$ , the latter result and the Poincaré inequality let us ensure that  $F_n \rightarrow F$  in  $L^2(0, T; V)$ . Since  $h_2 > 0$ , it follows in particular that

$$\lambda_{f,n} \rightarrow \lambda_f \text{ in } L^2(0, T; H).$$

**Existence of  $\mathcal{C} \subset W(0, T) \times L^2(0, T; (H^1(\Omega)))$  such that  $\mathcal{F}(\mathcal{C}) \subset \mathcal{C}$ .**

We aim now to prove that there exists a nonempty bounded closed convex set of  $W(0, T) \times L^2(0, T; H^1(\Omega))$ , denoted by  $\mathcal{C}$ , such that  $\mathcal{F}(\mathcal{C}) \subset \mathcal{C}$ .

We notice that this result will imply that there exists a real number  $\tilde{M} > 0$ , depending only on the initial data, such that for  $(\lambda_i, \lambda_f) = \mathcal{F}(\bar{\lambda}_i, \bar{\lambda}_f) \in W$ , we have

$$\|\nabla \lambda_i\|_{(L^2(0, T; H))^2} \leq \tilde{M} \quad \text{and} \quad \|\nabla \lambda_f\|_{(L^2(0, T; H))^2} \leq \tilde{M}. \quad (4.17)$$

Taking  $\lambda_i \in L^2(0, T; V)$  (resp.  $\alpha \lambda_f \in L^2(0, T; V)$ ) in (4.8) (resp. in (4.9)) and adding the two resulting equations lead to:

$$\begin{aligned} & \frac{\phi}{2} \int_{\Omega} \lambda_i^2 + \delta \int_{\Omega_T} |\nabla \lambda_i|^2 + \int_{\Omega_T} \alpha K(x) T_s(h) |\nabla \lambda_i|^2 + \alpha \int_{\Omega_T} K(x) h_2 |\nabla \lambda_f|^2 = \\ & \alpha \int_{\Omega_T} K(x) T_s(h) \nabla \lambda_i \cdot \nabla \lambda_f + \underbrace{\alpha \int_{\Omega_T} K(x) T_s(h) \nabla \lambda_i \cdot \nabla \bar{\lambda}_f}_{(1)} - \underbrace{\int_{\Omega_T} K(x) (\nabla f - \alpha \nabla h) \cdot \nabla \bar{\lambda}_i \lambda_i}_{(2)} \\ & - \underbrace{\alpha \int_{\Omega_T} K(x) \nabla h \cdot \nabla \bar{\lambda}_f \lambda_i}_{(3)} + \underbrace{\int_{\Omega_T} (-h + h_{obs}) \lambda_i - \alpha \int_{\Omega_T} (f - f_{obs}) \lambda_f}_{(4)} \end{aligned}$$

$\implies$  (since  $\delta_1 \leq h \leq h_2$ )

$$\begin{aligned} & \frac{\phi}{2} \int_{\Omega} \lambda_i^2 + \delta \int_{\Omega_T} |\nabla \lambda_i|^2 + \frac{\alpha h_2}{2} \int_{\Omega_T} K(x) |\nabla \lambda_f|^2 + \frac{\alpha}{2} \int_{\Omega_T} K(x) T_s(h) |\nabla \lambda_i|^2 \\ & \leq (1) + (2) + (3) + (4). \end{aligned}$$

But

$$\begin{aligned} |(1)| & \leq \frac{\alpha}{2} \int_{\Omega_T} K(x) T_s(h) |\nabla \lambda_i|^2 + \frac{\alpha K_+ h_2}{4} \int_{\Omega_T} |\nabla \lambda_i|^2 + \frac{\alpha}{3} \int_{\Omega_T} K(x) T_s(h) |\nabla \bar{\lambda}_f|^2, \\ |(2)| & \leq K_+ C_G C_4 (1 + \alpha) \left( \max_{t \in (0, T)} \int_{\Omega} \lambda_i^2 \right)^{1/4} \left( \int_{\Omega_T} |\nabla \lambda_i|^2 \right)^{1/4} \left( \int_{\Omega_T} |\nabla \bar{\lambda}_i|^2 \right)^{1/2} \\ & \leq \frac{K_+^2 C_G^2 C_4^2 (1 + \alpha)^2}{2 \epsilon_0} \left( \max_{t \in (0, T)} \int_{\Omega} \lambda_i^2 \right)^{1/2} \left( \int_{\Omega_T} |\nabla \lambda_i|^2 \right)^{1/2} + \frac{\epsilon_0}{2} \int_{\Omega_T} |\nabla \bar{\lambda}_i|^2, \\ & \leq \frac{K_+^4 C_G^4 C_4^4 (1 + \alpha)^4}{4 \epsilon_0 \epsilon_1} \left( \max_{t \in (0, T)} \int_{\Omega} \lambda_i^2 \right) + \frac{\epsilon_1}{2} \int_{\Omega_T} |\nabla \lambda_i|^2 + \frac{\epsilon_0}{2} \int_{\Omega_T} |\nabla \bar{\lambda}_i|^2, \\ |(3)| & \leq \frac{K_+^2 C_G^2 C_4^2 \alpha^4}{4 \epsilon_2 \epsilon_3} \left( \max_{t \in (0, T)} \int_{\Omega} \lambda_i^2 \right) + \frac{\epsilon_2}{2} \int_{\Omega_T} |\nabla \lambda_i|^2 + \frac{\epsilon_3}{2} \int_{\Omega_T} K(x) |\nabla \bar{\lambda}_f|^2, \end{aligned}$$

finally, using Poincaré's inequality (with constant  $C_P$ ) yields

$$|(4)| \leq \frac{\phi}{4} \left( \max_{t \in (0, T)} \int_{\Omega} \lambda_i^2 \right) + \frac{\alpha^2 \epsilon_4}{2} \int_{\Omega_T} K(x) |\nabla \lambda_f|^2 + \frac{1}{\phi} \int_{\Omega_T} (h - h_{obs})^2 + \frac{C_P^2}{2 \epsilon_4 K_-} \int_{\Omega_T} (f - f_{obs})^2.$$

By choosing  $\epsilon_0 = \frac{5\delta}{6}$ ,  $\epsilon_1 = \epsilon_2 = \frac{\delta}{4}$ ,  $\epsilon_3 = \frac{(1-5\alpha)\alpha h_2}{6}$ ,  $\epsilon_4 = h_2$ , we obtain

$$\begin{aligned} & \underbrace{\left( \frac{\phi}{4} - \frac{6K_+^4 C_G^4 C_4^4 (1+\alpha)^4}{5\delta^2} - \frac{6K_+^2 C_G^4 C_4^4 \alpha^3}{(1-5\alpha)\alpha h_2 \delta} \right)}_{\Theta_1} \left( \max_{t \in (0, T)} \int_{\Omega} \lambda_i^2 \right) \\ & + \left( \frac{\delta}{4} - \frac{\alpha K_+ h_2}{4} \right) \int_{\Omega_T} K(x) |\nabla \lambda_i|^2 + \frac{(1-\alpha)\alpha h_2}{2} \int_{\Omega_T} K(x) |\nabla \lambda_f|^2 + \frac{\delta}{2} \int_{\Omega_T} |\nabla \lambda_i|^2 \\ & \leq \frac{5}{12} \left( \delta \int_{\Omega_T} |\nabla \bar{\lambda}_i|^2 + (1-\alpha)\alpha h_2 \int_{\Omega_T} K(x) |\nabla \bar{\lambda}_f|^2 \right) \\ & \quad + \frac{1}{\phi} \int_{\Omega_T} (h - h_{obs})^2 + \frac{C_P^2}{h_2 K_-} \int_{\Omega_T} (f - f_{obs})^2. \end{aligned}$$

We assume that the coefficient  $\phi$  is big enough in order to ensure the nonnegativity of  $\Theta_1$ , namely

$$\phi - \frac{24K_+^4 C_G^4 C_4^4 (1+\alpha)^4}{5\delta^2} - \frac{24K_+^2 C_G^4 C_4^4 \alpha^3}{(1-5\alpha)\alpha h_2 \delta} > 0, \quad (4.18)$$

and that the hydraulic conductivity satisfies (2.7) given in Theorem 2.2:

$$K_+ \leq \frac{\delta}{\alpha h_2}.$$

We then choose the constant  $M > 0$  such that

$$M \geq 12 \times \left( \frac{1}{\phi} \int_{\Omega_T} (h - h_{obs})^2 + \frac{C_P^2}{2h_2 K_-} \int_{\Omega_T} (f - f_{obs})^2 \right), \quad (4.19)$$

hence we can conclude that

$$\begin{aligned} & \frac{\Theta_1}{4} \left( \max_{t \in (0, T)} \int_{\Omega} \lambda_i^2 \right) + \frac{1}{2} \int_{\Omega_T} (\delta |\nabla \lambda_i|^2 + \alpha(1-\alpha)h_2 K(x) |\nabla \lambda_f|^2) \\ & \leq \frac{5}{12} \int_{\Omega_T} (\delta |\nabla \bar{\lambda}_i|^2 + \alpha(1-\alpha)h_2 K(x) |\nabla \bar{\lambda}_f|^2) + \frac{M}{12}. \end{aligned} \quad (4.20)$$

If  $\int_{\Omega_T} (\delta |\nabla \bar{\lambda}_i|^2 + \alpha(1-\alpha)h_2 K(x) |\nabla \bar{\lambda}_f|^2) \leq M$  then (4.20) implies that

$$\int_{\Omega_T} (\delta |\nabla \lambda_i|^2 + \alpha(1-\alpha)h_2 K(x) |\nabla \lambda_f|^2) \leq M.$$

Setting  $\mathcal{C}$  the nonempty bounded closed convex set of  $W(0, T) \times W(0, T)$  defined by:

$$\begin{aligned} \mathcal{C} = \{ & (g_i, g_f) \in W(0, T) \times W(0, T); g_i(0) = 0, g_f(0) = 0, \\ & \int_{\Omega_T} (\delta |\nabla g_i|^2 + \alpha(1-\alpha)h_2 K(x) |\nabla g_f|^2) \leq M \}, \end{aligned}$$

we established that  $\mathcal{F}(\mathcal{C}) \subset \mathcal{C}$ . Then  $\mathcal{C}$  is a nonempty closed convex bounded set of  $L^2(0, T; H)^2$ , defined such that  $\mathcal{F}(\mathcal{C}) \subset \mathcal{C}$ . Since  $\mathcal{C}$  is also a bounded set in  $W(0, T) \times L^2(0, T; H^1(\Omega))$ , we also proved that  $\mathcal{F}$  restricted to  $\mathcal{C}$  is sequentially continuous in  $(L^2(0, T; H))^2$ . For the fixed point strategy, it remains to show the compactness of  $\mathcal{F}(\mathcal{C})$ . Since we work in metric spaces, proving its sequential compactness is sufficient. The compactness of  $\mathcal{F}_1(\mathcal{C})$  is straightforward

due to the Aubin's theorem. Let us further detail the proof for  $\mathcal{F}_2(\mathcal{C})$ . Let  $\{\lambda_{f,n}\}$  be a sequence in  $\mathcal{F}_2(\mathcal{C})$ . It is associated with a sequence  $\{(\bar{\lambda}_{i,n}, \bar{\lambda}_{f,n})\}$  in  $\mathcal{C}$ . The Aubin's compactness theorem let us ensure that there exists a subsequence, not renamed for convenience, and  $\bar{\lambda}_i \in W(0, T)$  such that  $\bar{\lambda}_{i,n} \rightarrow \bar{\lambda}_i$  in  $L^2(0, T; H)$  and almost everywhere in  $\Omega_T$ . Thus we can follow the lines beginning just after (4.12) for proving that  $\lambda_{f,n} \rightarrow \lambda_f$  in  $L^2(0, T; H)$ . The sequential compactness of  $\mathcal{F}_2(\mathcal{C})$  in  $L^2(0, T; H)$  is proved. The Schauder theorem allows us to conclude that there is a couple  $(\lambda_i, \lambda_f) \in \mathcal{C}$  such that  $\mathcal{F}(\lambda_i, \lambda_f) = (\lambda_i, \lambda_f)$ . This fixed point is a weak solution of the problem (4.6)-(4.7).

**Uniqueness of the solution of the adjoint problem:**

Let  $(\lambda_i, \lambda_f) \in W(0, T)^2$  and  $(\bar{\lambda}_i, \bar{\lambda}_f) \in W(0, T)^2$  be two solutions of (2.9). We set  $u = \lambda_i - \bar{\lambda}_i$  and  $v = \lambda_f - \bar{\lambda}_f$ . Clearly, since the system is linear,  $(u, v)$  satisfies (2.9) but the right member of (2.9) is equal to zero.

The previous estimates lead to the following inequality

$$\begin{aligned} & \frac{\Theta_1}{4} \left( \max_{t \in (0, T)} \int_{\Omega} u^2 \right) + \frac{1}{2} \int_{\Omega_T} (\delta |\nabla u|^2 + \alpha(1 - \alpha) h_2 K(x) |\nabla v|^2) \\ & \leq \frac{5}{12} \int_{\Omega_T} (\delta |\nabla u|^2 + \alpha(1 - \alpha) h_2 K(x) |\nabla v|^2), \end{aligned}$$

then

$$\left( \max_{t \in (0, T)} \int_{\Omega} u^2 \right) \leq 0, \quad \int_{\Omega_T} |\nabla u|^2 \leq 0 \quad \text{and} \quad \int_{\Omega_T} |\nabla v|^2 \leq 0.$$

Since  $(u, v) \in L^2(0, T; H_0^1(\Omega))^2$ , this implies  $u = 0$  a.e. in  $\Omega_T$  and  $v = 0$  a.e. in  $\Omega_T$ .

The condition (4.18) may look very restrictive. However, we can again pick the coefficient  $\phi$  arbitrary large (for it corresponds to a time scaling), so that the conditions (4.18) can indeed be satisfied. Setting

$$t_0 = \frac{T}{24 K_+^2 C_G^4 C_4^4} \left( \frac{K_+^2 (1 + \alpha)^4}{5 \delta^2} + \frac{\alpha^3}{(1 - 5\alpha) \alpha h_2 \delta} \right)^{-1},$$

we proved the existence and the uniqueness for the short time  $t \in [0, t_0]$ . Taking  $t = t_0$  as new initial data, the existence and uniqueness is obtained for all  $t_0 \leq t \leq 2t_0$ . Using this observation inductively, we derive the result on the whole range of study  $[0, T]$ . This ends the proof of Theorem 2.2.

**Remark 3** The condition (2.7) is of same type of that implied in Theorem 3.2 giving the uniqueness. Since  $\alpha = 0.0125$ ,  $\delta = \mathcal{O}(1)$  and  $h_2 = \mathcal{O}(10)$ , the condition (2.7) is not so restrictive.

#### 4.4 Optimality conditions

To characterize a solution to the optimization problem  $(\mathcal{O})$ , the gradient of the cost function is calculated using the Lagrangian defined by (4.4). An essential step is to prove that the operator associating to the hydraulic conductivity  $K$  the solution,  $(f(K), h(K))$  of (2.1)-(2.4), is continuous and differentiable. The main point is to find the appropriate function spaces for the implicit function theorem to be applicable. The regularity results established for  $(f(K), h(K))$  in Theorem 3.1 and the following Proposition 4.1 allow to conclude.

##### 4.4.1 Proof of Theorem 2.3

The first results regarding these additional regularity is given in the following proposition

**Proposition 4.1** Let  $(f(K), h(K))$  the solution of (2.1)-(2.4) associated with the hydraulic conductivity  $K \in U_{adm}$ . For all  $(F_i, F_f) \in (L^2(0, T; H^{-1}(\Omega)))^2$ , the following problem a) Find  $(\lambda_i, \lambda_f) \in (W(0, T) \cap L^\infty(0, T; L^2(\Omega)))^2$  such that  $\lambda_i(0, \cdot) = 0$  and

$$\int_{\Omega_T} [\phi \langle \frac{\partial \lambda_i}{\partial t}, \varphi_i \rangle + (\delta + \alpha(h_2 - h)K(x)) \nabla \lambda_i \cdot \nabla \varphi_i - (h_2 - h)K(x) \nabla \lambda_f \cdot \nabla \varphi_i] dx dt \quad (4.21)$$

$$+ \int_{\Omega_T} \lambda_i K(x) [\nabla f \cdot \nabla \varphi_i - \alpha \nabla h \cdot \nabla \varphi_i] dx dt = \int_{\Omega_T} F_i \varphi_i dx dt, \quad \forall \varphi_i \in L^2(0, T; H_0^1(\Omega)),$$

$$\int_{\Omega_T} K(x) [h_2 \nabla \lambda_f \cdot \nabla \varphi_f - \alpha(h_2 - h) \nabla \lambda_i \cdot \nabla \varphi_f] dx dt \quad (4.22)$$

$$+ \int_{\Omega_T} \alpha \lambda_i K(x) \nabla h \cdot \nabla \varphi_f dx dt = \int_{\Omega_T} F_f \varphi_f dx dt \quad \forall \varphi_f \in L^2(0, T; H_0^1(\Omega)),$$

has one and only one solution.

b) Let  $(\phi, h_2, K_+, \delta) \in (\mathbb{R}_*^+)^4$  satisfying (3.1), (3.3), (3.5) and (3.6) for  $r = \sigma > 4$ , ensuring  $(f(K), h(K)) \in L^\sigma(0, T; W^{1,\sigma}(\Omega))^2$ . There exists  $s \in ]2, \sigma[$ , such that, if  $(F_i, F_f) \in L^s(0, T; W^{-1,s}(\Omega))$  and  $(\lambda_i(0, \cdot), \lambda_f(0, \cdot)) \in (W_0^{1,s}(\Omega))^2$ , then  $(\lambda_f, \lambda_i) \in L^s(0, T; W^{1,s}(\Omega))$ .

*Proof:* The proof of a) of Proposition (4.1) is quite similar to the proof of Theorem 2.2, for this reason, we omit it here.

Regarding point b), we use again the Meyers regularity result by remarking that, since  $(\lambda_i, \lambda_f) \in (W(0, T) \cap L^\infty(0, T; L^2(\Omega)))^2$  and  $((f - f_D, h - h_D) \in L^\sigma(0, T; W_0^{1,\sigma}(\Omega)))$ , we can find  $s \in ]2, \sigma[$  s.t.  $div(\lambda_i \nabla h)$  belongs to  $L^s(0, T; W^{-1,s}(\Omega))$  (and the same for terms with  $f$ ). Indeed  $\lambda_i(t, \cdot) \in H_0^1(\Omega) \subset L^p(\Omega)$ ,  $\forall p \in [1, \infty[$  and  $h - h_D \in L^\sigma(0, T; W_0^{1,\sigma}(\Omega))$ , with  $\sigma > 4$ , then

$$\int_0^T \int_\Omega |\lambda_i|^s |\nabla h|^s \leq \|\nabla h\|_{L^\sigma(\Omega_T)}^s \left( \int_0^T \int_\Omega |\lambda_i|^{s\sigma/(\sigma-s)} dx dt \right)^{\frac{\sigma-s}{s}}.$$

If  $s > 2$  is chosen s.t.  $\frac{s\sigma}{(\sigma-s)} = 4$ , it means  $s = \frac{4\sigma}{4+\sigma}$  (which is  $> 2$  if  $\sigma > 4$ ), we get

$$\int_0^T \int_\Omega |\lambda_i|^s |\nabla h|^s \leq \|\nabla h\|_{L^\sigma(\Omega_T)}^s \|\lambda_i\|_{L^\infty(0,T;L^2(\Omega))}^{2(\sigma-s)/s} \|\lambda_i\|_{L^2(0,T;H_0^1(\Omega))}^{2(\sigma-s)/s}.$$

We directly treat the other terms of (4.23) since functions  $h$  belongs to  $L^\infty(\Omega_T)$ .  $\square$

To prove the differentiability of the operator  $\mathcal{Q}$ , we introduce an application  $\mathcal{R}$ , defined as a function of  $(f(K), h(K), K)$ . The equality (4.25) below giving  $\mathcal{R}$  allows to implicitly define  $\mathcal{Q}$ . By applying the implicit function theorem to  $\mathcal{R}$ , we then establish the differentiability of  $\mathcal{Q}$ . So, we consider the mapping  $\mathcal{R}$  such that

$$\begin{aligned} \mathcal{R} : Z(0, T) \times \text{Int}(U) &\longrightarrow (L^2(0, T; H^{-1}(\Omega)))^2 \\ (\bar{f}, \bar{h}, K) &\longrightarrow \mathcal{R}(\bar{f}, \bar{h}, K) \end{aligned}$$

where  $Z(0, T)$  (resp.  $U$ ) is defined by (2.10) (resp. by (2.10)) and  $(\bar{f}, \bar{h}) = (f - f_D, h - h_D)$ . We note that  $U_{adm} \subset \text{Int}(U)$ , 'Int' denoting the interior of  $U$  for the topology of  $BV(\Omega) \cap L^\infty(\Omega)$  (cf. [21]).

Then,  $\forall (\varphi_f, \varphi_i) \in (L^2(0, T; H_0^1(\Omega)))^2$ , we define  $\mathcal{R}$  as follows

$$\langle \mathcal{R}(\bar{h}_1, \bar{h}, K), (\varphi_f, \varphi_i) \rangle = \int_{t_0}^T \int_\Omega \phi \langle \frac{\partial h}{\partial t}, \varphi_i \rangle_{V', V} dt$$

$$\begin{aligned}
& + \int_0^T \int_{\Omega} (\delta + \alpha K(x) (h_2 - h)) \nabla h \cdot \nabla \varphi_i \, dxdt - \int_0^T \int_{\Omega} K(x) (h_2 - h) \nabla f \cdot \nabla \varphi_i \, dxdt \\
& + \int_0^T \int_{\Omega} h_2 K(x) \nabla f \cdot \nabla \varphi_f \, dxdt - \int_0^T \int_{\Omega} \alpha K(x) (h_2 - h) \nabla h \cdot \nabla \varphi_f \, dxdt \\
& + \int_0^T \int_{\Omega} Q_s \varphi_i \, dxdt - \int_0^T \int_{\Omega} (Q_s + Q_f) \varphi_f \, dxdt. \tag{4.23}
\end{aligned}$$

Before applying the implicit functions theorem to  $\mathcal{R}$ , we first aim to prove that  $\mathcal{R}$  is continuous and differentiable from  $Z(0, T)^2 \times \text{Int}(U)$  to  $(L^2(0, T; H^{-1}(\Omega)))^2$ . Since function  $h$  belongs to  $L^\infty(\Omega_T)$ , the continuity of  $\mathcal{R}$  is clear. Analogously,  $\mathcal{R}$  is continuously differentiable with respect to  $K$  on  $Z(0, T)^2 \times \text{Int}(U)$  and for all  $L \in \text{Int}(U)$

$$\begin{aligned}
\langle D_K \mathcal{R}(\bar{f}, \bar{h}, K) \cdot L, (\varphi_f, \varphi_i) \rangle & = \int_{\Omega_T} L (h_2 - h) [\alpha \nabla h + \nabla f] \cdot \nabla \varphi_i \, dxdt \\
& + \int_{\Omega_T} L [h_2 \nabla f + \alpha (h_2 - h) \nabla h] \cdot \nabla \varphi_f \, dxdt. \tag{4.24}
\end{aligned}$$

Regarding  $D_{(\bar{f}, \bar{h})} \mathcal{R}(\bar{f}, \bar{h}, K)$ , we first check the differentiability of  $\mathcal{R}$  with respect to  $(\bar{f}, \bar{h})$ . To this end, we define,  $\forall (\lambda_f, \lambda_i) := \lambda \in (Z(0, T))^2$

$$\begin{aligned}
\langle D_{(\bar{f}, \bar{h})} \mathcal{R}(\bar{f}, \bar{h}, K) \cdot \lambda, (\varphi_f, \varphi_i) \rangle & = \int_0^T \phi \left\langle \frac{\partial \lambda_i}{\partial t}, \varphi_i \right\rangle_{V', V} dt \\
& + \int_{\Omega_T} [(\delta + \alpha (h_2 - h) K(x)) \nabla \lambda_i \cdot \nabla \varphi_i - (h_2 - h) K(x) \nabla \lambda_f \cdot \nabla \varphi_i] \, dxdt \\
& + \int_{\Omega_T} \lambda_i [K(x) \nabla f \cdot \nabla \varphi_i - \alpha K(x) \nabla h \cdot \nabla \varphi_i] \, dxdt + \int_{\Omega_T} \alpha \lambda_i K(x) \nabla h \cdot \nabla \varphi_f \, dxdt \\
& + \int_{\Omega_T} K(x) [h_2 \nabla \lambda_f \cdot \nabla \varphi_f - \alpha (h_2 - h) \nabla \lambda_i \cdot \nabla \varphi_f] \, dxdt,
\end{aligned}$$

So, a direct calculation gives

$$\begin{aligned}
& \langle \mathcal{R}(\bar{f} + \lambda_f, \bar{h} + \lambda_i, K) - \mathcal{R}(\bar{f}, \bar{h}, K) - D_{(\bar{h}_1, \bar{h})} \mathcal{R}(\bar{h}_1, \bar{h}, K) \cdot \lambda, (\varphi_i, \varphi_f) \rangle = \\
& - \int_{\Omega_T} K(x) (\lambda_i (\alpha \nabla \lambda_i + \nabla \lambda_f) \cdot \nabla \varphi_i + \lambda_f (\alpha \nabla \lambda_i + \nabla \lambda_f) \cdot \nabla \varphi_f) \, dxdt := \Delta_{f,i}.
\end{aligned}$$

Then

$$\begin{aligned}
|\Delta_{f,i}| & \leq \kappa_M \|\lambda_i\|_{L^{2s/(s-2)}(\Omega_T)} (\alpha \|\nabla \lambda_i\|_{L^s(\Omega_T)} + \|\nabla \lambda_f\|_{L^s(\Omega_T)}) \|\nabla \varphi_i\|_{L^2(\Omega_T)} \\
& + \kappa_M \|\lambda_f\|_{L^{2s/(s-2)}(\Omega_T)} (\alpha \|\nabla \lambda_i\|_{L^s(\Omega_T)} + \|\nabla \lambda_f\|_{L^s(\Omega_T)}) \|\nabla \varphi_f\|_{L^2(\Omega_T)}. \tag{4.25}
\end{aligned}$$

Setting  $l \in \{i, f\}$ , we have

$$\int_{\Omega_T} \lambda_l^{2s/(s-2)} \leq \int_0^T (\|\lambda_l\|_{L^\infty(\Omega)}^{4/(s-2)} \times \int_{\Omega} \lambda_l^2) dt \leq \max_{t \in ]0, T[} \int_{\Omega} \lambda_l^2 \times \int_0^T \|\lambda_l\|_{L^\infty(\Omega)}^{4/(s-2)} dt.$$

We take  $s \in ]2, 4[$  such that  $\frac{4}{s-2} = s$  (i.e.  $s = 1 + \sqrt{5}$ ), since  $W^{1,s}(\Omega) \subset L^\infty(\Omega)$ , we get

$$\int_{\Omega_T} \lambda_l^{2s/(s-2)} \leq C \|\lambda_l\|_{L^\infty(0, T; L^2(\Omega))}^2 \|\nabla \lambda_l\|_{L^s(\Omega_T)}^s, \quad l = i, f.$$

Doing the same with the other terms of (4.25), we prove the differentiability of  $\mathcal{R}$  with respect to  $(\bar{f}, \bar{h})$  on  $Z(0, T) \times \text{Int}(U)$  at the point  $(\bar{f}, \bar{h}, K)$  (where  $(\bar{f}(K), \bar{h}(K))$  is solution of (2.1)-(2.2)). Moreover, Proposition 4.1 ensures that  $D_{(\bar{f}(K), \bar{h}(K))} \mathcal{R}$  is an isomorphism from  $Z(0, T)$

to  $(L^2(0, T; H^{-1}(\Omega)))^2$ ,  $\forall K \in U_{adm}$ . The requirements of implicit function theorem are satisfied.  $\square$

Since function  $h$  belongs to  $L^\infty(\Omega_T)$ , the continuity of  $\mathcal{R}$  is clear. Analogously,  $D_K \mathcal{R}(\bar{f}, \bar{h}, K)$  is continuously differentiable on  $Z(0, T) \times Int(U)$ . Regarding  $D_{(\bar{f}, \bar{h})} \mathcal{R}(\bar{f}, \bar{h}, K)$ , we first notice that it belongs to the space  $\mathcal{L}((Z(0, T))^2, (L^2(0, T; H^{-1}(\Omega))))^2$ , indeed  $\forall (\varphi_i, \varphi_f) \in L^2(0, T; H_0^1(\Omega))^2$

$$\begin{aligned} | \langle D_{(\bar{h}_1, \bar{h})} \mathcal{R}(\bar{h}_1, \bar{h}, K) \cdot \lambda, (\varphi_i, \varphi_f) \rangle | &\leq \phi \|\lambda_i\|_{L^2(0, T; H^{-1})} \|\varphi_i\|_{L^2(0, T; H_0^1)} \\ &+ (\delta + \alpha K_+ h_2) \|\nabla \lambda_i\|_{L^2(\Omega_T)} \|\nabla \varphi_i\|_{L^2(\Omega_T)} + K_+ h_2 \|\nabla \lambda_f\|_{L^2(\Omega_T)} \|\nabla \varphi_i\|_{L^2(\Omega_T)} \\ &+ K_+ h_2 \|\nabla \lambda_f\|_{L^2(\Omega_T)} \|\nabla \varphi_f\|_{L^2(\Omega_T)} + \alpha K_+ h_2 \|\nabla \lambda_i\|_{L^2(\Omega_T)} \|\nabla \varphi_f\|_{L^2(\Omega_T)} \\ &+ K_+ \|\lambda_i\|_{L^{2s/(s-2)}(\Omega_T)} \|\nabla f\|_{L^s(\Omega_T)} \|\nabla \varphi_i\|_{L^2(\Omega_T)} \\ &+ \alpha K_+ \|\lambda_i\|_{L^{2s/(s-2)}(\Omega_T)} \|\nabla h\|_{L^s(\Omega_T)} (\|\nabla \varphi_i\|_{L^2(\Omega_T)} + \|\nabla \varphi_f\|_{L^2(\Omega_T)}). \end{aligned}$$

Besides

$$\int_{\Omega_T} \lambda_i^{2s/(s-2)} \leq \int_0^T \|\lambda_i\|_{L^\infty}^{4/(s-2)} \times \int_{\Omega} \lambda_i^2 dt \leq \max_{t \in (0, T)} \int_{\Omega} \lambda_i^2 \times \int_0^T \|\lambda_i\|_{L^\infty}^{4/(s-2)} dt.$$

We take  $s \in ]2, 4[$  such that  $\frac{4}{s-2} = s$  (i.e.  $s = 1 + \sqrt{5}$ ), since  $W^{1, s}(\Omega) \subset L^\infty(\Omega)$ , we get

$$\int_{\Omega_T} \lambda_i^{2s/(s-2)} \leq C \|\lambda_i\|_{L^\infty(0, T; L^2(\Omega))}^2 \|\nabla \lambda_i\|_{L^s(\Omega_T)}^s.$$

Using the same kind of estimates, we prove the continuity of  $D_{(\bar{f}, \bar{h})} \mathcal{R}(\bar{f}, \bar{h}, K)$  on  $Z(0, T) \times Int(U)$  at the point  $(\bar{f}, \bar{h}, K)$  (where  $(\bar{f}(K), \bar{h}(K))$  is solution of (2.1)-(2.4)). Moreover, Proposition 4.1 ensures that  $D_{(\bar{f}(K), \bar{h}(K))} \mathcal{R}(\bar{f}, \bar{h}, K)$  is an isomorphism from  $Z(0, T)$  to  $L^2(0, T; H^{-1}(\Omega))$ , for all  $K \in U_{adm}$ .

The requirements of implicit function theorem are satisfied, then Proposition ?? is proved.

#### 4.4.2 Proof of Theorem 2.4

The solution corresponds to a saddle point of the Lagrangian considered as a function of the independent variables  $h, f, \lambda_i, \lambda_f$  and  $K$  with  $\lambda_i$  and  $\lambda_f$  the Lagrange multipliers. The required minimum,  $K^*$ , satisfies optimality system (4.5).

Thanks to Theorem 2.3, the application  $K \rightarrow \mathcal{Q}(K) = (f(K), h(K))$  implicitly defined by the direct problem (2.1)-(2.4), is differentiable.

So the application  $K \rightarrow \mathcal{J}(K) = \mathcal{L}(K, f(K), h(K), \lambda_i, \lambda_f)$  is differentiable with respect to  $(K, f, h)$  and

$$D_K \mathcal{J}(K^*) \cdot L = \partial_K \mathcal{L}(K^*, f(K^*), h(K^*), \lambda_i, \lambda_f) \cdot L + \partial_{(h, f)} \mathcal{L}(K^*, f(K^*), h(K^*), \lambda_i, \lambda_f) \cdot \mathcal{H},$$

where the Lagrangian  $\mathcal{L}$  is defined by (4.4) and  $\mathcal{H} = \frac{d\mathcal{Q}}{dK}(K^*) \cdot L = (\varphi_i, \varphi_f)$ . Furthermore, every minimum of  $\mathcal{J}$  on  $U_{adm}$  verifies (2.12).

Besides, the gradient of the cost function is given by,  $\forall \delta_K \in U_{adm}$

$$\begin{aligned} D_K \mathcal{J}(K^*) \cdot \delta_K &= \int_{\Omega_T} \delta_K [(h_2 - h^*)(\alpha \nabla h^* - \nabla f^*) \cdot \nabla \lambda_i + (h_2 \nabla f^* - \alpha (h_2 - h^*) \nabla h^*) \cdot \nabla \lambda_f] \\ &+ \int_{\Omega_T} [-\phi \frac{\partial \lambda_i}{\partial t} \varphi_i + (\delta + \alpha K^* (h_2 - h^*)) \nabla \lambda_i \cdot \nabla \varphi_i - \alpha K^* (h_2 - h^*) \nabla \lambda_f \cdot \nabla \varphi_i] \end{aligned}$$

$$\begin{aligned}
& + \int_{\Omega_T} K^* [\nabla f^* \cdot \nabla \lambda_i - \alpha \nabla h^* \cdot \nabla (\lambda_i - \lambda_f)] \varphi_i - \int_{\Omega_T} (h_{obs} - h^*) \varphi_i \, dxdt \\
& + \int_{\Omega_T} [K^* h_2 \nabla \lambda_f \cdot \nabla \varphi_f - K^* (h_2 - h^*) \nabla \lambda_i \cdot \nabla \varphi_f] - \int_{\Omega_T} (f_{obs} - f^*) \varphi_f
\end{aligned}$$

with  $h^* = h(K^*)$ ,  $f^* = f(K^*)$ . We deduce from Theorem 2.2, that the adjoint problem has a unique solution  $\lambda_i^* = \lambda_i(K^*, f^*, h^*)$ ,  $\lambda_f^* = \lambda_f(K^*, f^*, h^*)$ , then taking  $\lambda_i = \lambda_i^*$  and  $\lambda_f = \lambda_f^*$  in (4.26) leads to (2.12), which ends the proof.

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## 4.5 References

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