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An application of Nekhoroshev theory to the study of the perturbed hydrogen atom

Francesco Fassò*, Daniele Fontanari† and Dmitrií A. Sadovskii‡

(June 23, 2015)

Abstract

We return to the Keplerian or n -shell approximation to the hydrogen atom in the presence of weak static electric and magnetic fields. At the classical level, this is a Hamiltonian system with the phase space $S^2 \times S^2$. Its principal order Hamiltonian H_0 was known already to Pauli in 1926. H_0 defines an isochronous system with a linear flow on $S^2 \times S^2$ and with frequencies depending on the external fields. Small perturbations $H_0 + P$ due to higher order terms can be studied by further normalization, either resonant or nonresonant. We study the question, raised previously, of how to decide for given parameters of the fields what normalization should be used and with regard to which resonances. We base this analysis on the Nekhoroshev theory—a branch of the Hamiltonian perturbation theory that complements the Kolmogorov-Arnold-Moser theorem. Our answer depends on the *a priori* choice of the maximal order N of resonances that are going to be taken into account (a cutoff). For any given N , there is a decomposition of the parameter space into resonant and nonresonant zones, and a normal form with a remainder of order $\exp(-N)$ may be consistently constructed in each of such zones.

Keywords: Perturbed hydrogen atom; n -shell; Resonances, Normal form; Nekhoroshev theory.

MSC: 37J40, 70H08, 70F05, 81V45 , 37J40.

1 Introduction

A. Aim of the paper. In the ninety years since Pauli’s work [26], perturbation theory has been widely applied to the study of the hydrogen atom in weak static electric and magnetic fields; we refer to the recent review [10]. The disclosure of resonances [9, 31] has brought new interest to this field. While they represent a very interesting new subject of theoretical and experimental research, only a few of them have been investigated. It is not even understood if, and to which high order, resonances can be reached. Experimentalists would like to know the tolerances on the parameters of the system in order to be sure that the system is within a particular resonance. Since resonances depend on the parameters of the external fields, the idea has emerged that the parameter space should be decomposed into ‘zones’ of approximate resonance, and any system within a given zone should

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be approximated by the corresponding resonant normal form [9]. However, neither a criterion for the choice of the resonances which should be taken into consideration in the decomposition of the parameter space, nor general indications on the size of these regions were given. It was only suggested that “At the level of common physical intuition, it seems that the size and the stability of a zone decrease with the order of the resonance” ([10], section I.C). The purpose of the present article is that of providing some insight into these questions by approaching the problem in the realm of Nekhoroshev theory, which in our opinion is the right setting for its study.

B. The perturbed hydrogen atom. The study of the hydrogen atom in weak electric and magnetic fields goes back to Pauli [26] (see also [14, 1, 5, 35] and the recent review [10]). In this approach, the starting point is the specific Kepler Hamiltonian with a small perturbation, which represents the external fields. Due to the separation of time scales between the Kepler motion and the perturbation, the system is first averaged over the Kepler flow a suitably large number of times. Neglecting the remainder produces a truncated normal form which is in involution with the Keplerian integral (or action) n , that corresponds to the principal quantum number of the hydrogen atom. For each $n > 0$, reduction over the Keplerian symmetry of this truncated normal form produces an n -dependent Hamiltonian system with two degrees of freedom and phase space $S^2 \times S^2$, that we will call the *n-shell normal form*.

The n -shell normal form is nonintegrable, but its principal order, that may be called *Pauli Hamiltonian*, is a completely integrable Hamiltonian on $S^2 \times S^2$. Its flow consists of a uniform rotation on each of the two spheres, about an axis whose orientation depends on the external fields and with a frequency which is constant on the sphere and depends on n and on the external fields as well. Thus, the flow of the integrable part is isochronous and, depending on the ratio of the frequencies of the two rotations, it is either nonresonant, hence quasi-periodic, or resonant, hence periodic.

So far, the analysis of the n -shell normal form has been carried over for only a few resonances of the lowest orders, specifically, the exact 1 : 1 resonance (exactly orthogonal fields), a widely researched system, see [30, 6] and references therein as well as the review [10], the near 1:1 resonance (nearly orthogonal fields) [9, 31, 7, 10] and the near 2 : 1 resonance [10, 8]. For the collinear fields, many resonances, including the 1 : 0, have been uncovered in [29] without detailed analysis [10].

In each of these cases, the analysis was based on averaging over the integrable isochronous flow of the Pauli Hamiltonian, carried over a sufficiently high number of times so as to produce, after truncation of the remainder, a structurally stable normal form Hamiltonian. This ‘second’ truncated normal form is integrable and generates a foliation by invariant tori, of (generic) dimension two, whose topological properties were studied.

More precisely, for each of these few low order resonances, the normalization and truncation procedure was applied to approximately resonant cases, corresponding to external fields whose parameters fall in a certain ‘zone’ around the fields parameters that produce the exact resonance. In the approximately resonant case, the n -shell normal form was regarded not as a perturbation of its own integrable isochronous part, but as a perturbation of the integrable isochronous Hamiltonian that has the frequencies of the exact resonance nearby. The difference between the two frequencies was regarded as a small ‘parametric detuning’. The question raised in [9, 10] is how much can the detuning be increased before the resonant zone is left.

C. Nekhoroshev theory. The celebrated Nekhoroshev theorem on the stability over exponentially long times of nearly integrable Hamiltonian systems [23, 24, 2, 27] is to a large extent a theory of resonant motions of such systems. Its key ingredient is the decomposition of the frequency and action spaces into resonant and nonresonant zones.

The starting point of Nekhoroshev construction is the choice of a cutoff N —only resonances of order up to N are taken into consideration. This reflects the idea that, on a time scale T , only resonances of order up to a certain threshold N , that depends on T , produce effects which can

be detected on that time scale, while higher order resonant motions are indistinguishable, on that time scale, from nonresonant motions. For real analytic systems, due to the exponential decay of Fourier harmonics, the link between the cutoff and the time scale is

$$T_N \sim e^N. \quad (1)$$

For any value of N , frequencies suitably close to each exact resonance of order up to N form zones of approximate resonance and are treated as resonant, while all other frequencies form the nonresonant zone and are treated as nonresonant. Nekhoroshev theory produces a detailed, quantitative description of this decomposition, and proves the existence, in each (non)resonant zone, of a normal form adapted to the (non)resonance properties of the zone and with a remainder of the order of e^{-N} .

This construction is then used, under additional ('convexity' or 'steepness') hypotheses on the unperturbed Hamiltonian, to bound all motions on the time scale (1).

Finally, the last ingredient in Nekhoroshev theory is to choose the cutoff so as to maximize the stability time (1). This leads to $N(\epsilon) \sim \epsilon^{-\text{const}}$ and the resulting 'Nekhoroshev time scale' is therefore the celebrated 'exponentially long' time scale

$$T_{N(\epsilon)} \sim e^{N(\epsilon)} \sim e^{1/\epsilon^{\text{const}}}. \quad (2)$$

This construction has some implications which may at first sight appear somewhat counterintuitive. Since decreasing ϵ increases $N(\epsilon)$, more and more resonant zones appear in the decomposition as $\epsilon \rightarrow 0$; frequencies which were treated as nonresonant for a given ϵ might be treated as resonant for a smaller ϵ , or vice versa. The reason behind this is that decreasing ϵ increases the time scale and higher-order resonances become important. Of course, any given exact resonance will eventually end inside a resonant zone as $\epsilon \rightarrow 0$.

It should be noted that, while Nekhoroshev approach is relevant to classical systems, not all of it might be relevant to the semiclassical limit. This applies, particularly, to the appearing of time scales which go to infinity as $\epsilon \rightarrow 0$, that might be a dubious element in the description of quantum systems.

D. Content of the paper. In this paper we study the n -shell normal form Hamiltonian via Nekhoroshev theory. Given that our interest is not the dynamics of the system, and in view of the unclear role of long time scales in quantum mechanics, especially on a given compact n -shell phase space, we will not try to maximize the time scales on which the results apply and to reach the Nekhoroshev time scale (2). Instead, we will consider the cutoff N as a free parameter, independent of the size of the perturbation. Its role will only be that of determining the set of resonances that are taken into consideration.¹

A difference between our approach and the standard Nekhoroshev theory is that the latter is mainly addressed to anisochronous systems. In Nekhoroshev theory, it is the action space that needs to be decomposed into resonant and nonresonant zones, and the effort is to control motions through the resulting network of zones—e.g. by proving that motions that start inside one of these zones do not leave it on the considered time scale. Here, instead, we will consider perturbations of an isochronous system, whose frequencies depend on the parameters of the external fields. Accordingly, it will be the parameter space that will be decomposed into resonant and nonresonant zones. From this point of view, our treatment resembles the formulation of Nekhoroshev theory for elliptic equilibria of Hamiltonian systems [13, 25, 28], with the simplification that there will be no need to prove the confinement of motions.

Our approach introduces some novelties to the previous studies of the perturbed hydrogen atom. We will study the n -shell normal form not for fixed values of the resonances, and hence of the fields

¹Of course, in the problem at hand, if the order of the n -shell normal form is fixed, it is meaningless to increase the cutoff N beyond a certain threshold.

parameters, but—as is typical of Nekhoroshev approach—for all values of the parameters. In doing so we will also consider nonresonant normal forms, which have not been regarded as particularly interesting in the mentioned references, presumably because of the triviality of the topology of the fibration by the two-dimensional invariant tori they produce.² In this way we will obtain an N -dependent decomposition of the parameter space in resonant and nonresonant zones: around each exact resonance $\nu = (\nu_1, \nu_2) \in \mathbb{Z}^2$ of order $|\nu| = |\nu_1| + |\nu_2| \leq N$ there is a resonant zone of size of the order of

$$\frac{1}{N|\nu|}. \quad (3)$$

We will prove that, with this decomposition of the parameter space, a (non)resonant normal form can be consistently constructed for all systems in each (non)resonant zone, with a remainder of the order of e^{-N} .

Thus, in our approach, the question of which normal form to construct for a given system does not have a unique answer: the answer depends on the choice of the cutoff, that is, of the set of resonances that are going to be taken into consideration. Increasing N , more resonant zones appear in the decomposition and frequencies which were treated as nonresonant for a smaller N might now be treated as resonant, or vice versa.

From the point of view of Hamiltonian perturbation theory, the problem we study here has two related peculiarities. One is that the phase space is not the standard action-angle space, but a compact manifold, $S^2 \times S^2$. The other is that the foliation by the invariant tori of the integrable part, that have dimension two in case of nonresonance and dimension one in case of resonance, has singularities corresponding to equilibrium points on each sphere. Therefore, we cannot resort to the standard formulation of Hamiltonian perturbation theory, that employs action-angle coordinates, but we need to mix it with a Birkhoff-like normal form construction that employs Cartesian-like coordinates. Moreover, since we aim at giving a global result on the phase space, we cannot simply exclude from our consideration the singularities of the foliation by the invariant tori, but we need to construct the normal forms globally on $S^2 \times S^2$. We will achieve this goal by resorting to an atlas formed by a number of charts and using ideas and techniques from [12, 4, 13]. There are other problems of quantum-mechanical interest where the phase space is a product of spheres (see [15, 16]), and our treatment here may also serve as a reference for them.

We review the properties of the n -shell normal form in Section 2. In Section 3 we present the results of applying Nekhoroshev theory to the n -shell normal form (Proposition 2). The proof of Proposition 2, which is rather technical but is not strictly necessary for the comprehension of the entire argument, is relegated to Section 4.

2 The n -shell normal form

We describe here, very shortly, the n -shell normal form Hamiltonian, which is the starting point of our work. As already mentioned, this goes back to Pauli. For details, see e.g. [26, 35, 10, 31]. The first n -shell normal forms (the next term after the linear part) were, probably, obtained by Soloviev [32, 33] for the quadratic Zeeman effect (i.e., for purely magnetic field) and crossed fields [34], and in a number of studies that followed, such as [19, 20, 17], see more in [10].

The hydrogen atom is modelled as a classical Keplerian system formed by an electron in the Coulomb field created by an infinitely massive proton. In atomic units, the Hamiltonian can be written as

$$\frac{1}{2}\|p\|^2 - \frac{1}{\|q\|} - \varepsilon \cdot q - \frac{1}{2}\mathcal{B} \cdot q \times p + \frac{1}{8}\|\mathcal{B} \times q\|^2, \quad q \in \mathbb{R}^3 \setminus \{0\}, p \in \mathbb{R}^3. \quad (4)$$

²A Nekhoroshev approach which employs only resonant normal forms, and hence leads to a decomposition of the parameter space into resonant zones alone, might be based on Lochak's version of Nekhoroshev theory [21].

Here and in the following we denote by $\|\cdot\|$ the Euclidean norm and by a dot the Euclidean scalar product in \mathbb{R}^p , $p \geq 2$, and by a cross the vector product in \mathbb{R}^3 . Let $E < 0$ be the energy of the Kepler system, and denote by n the Keplerian action and the corresponding principal atomic quantum number

$$n = (-2E)^{-1/2}.$$

Assuming that the ratios between the electric and magnetic fields and (certain powers of) $-E$ are small it is possible to average Hamiltonian (4) any number of times over the Keplerian flow. After truncation of the remainder, this gives a two-degrees of freedom Hamiltonian system with phase space diffeomorphic to $S^2 \times S^2$ and Hamiltonian

$$H_0 + \epsilon P, \quad (5)$$

that we call the n -shell normal form and now describe.

The parameter ϵ in (5) is

$$\epsilon = (\|\mathcal{B}_n\|^2 + \|\mathcal{E}_n\|^2)^{1/2}$$

with $\mathcal{B}_n = n^3 \mathcal{B}$ and $\mathcal{E}_n = 3n^4 \mathcal{E}$.

The phase space of system (5) is $S_{n/2}^2 \times S_{n/2}^2$, where S_r^2 is the two-sphere of radius $r > 0$ in \mathbb{R}^3 , with the symplectic structure given by the sum of (the pullbacks under the projections onto the two factors of) the normalized area element on each sphere. The normalized area element σ_r on the sphere $S_r^2 \subset \mathbb{R}^3 \ni x$ is

$$\sigma_r(x)(u, v) = r^{-2} x \cdot u \times v \quad \forall x \in S_r^2, u, v \in T_x S_r^2$$

and, correspondingly, the Hamiltonian vector field of a function $H : S^2 \rightarrow \mathbb{R}$ is $-x \times \text{grad}H(x)$; in particular, the Hamiltonian vector field of the linear Hamiltonian $H(x) = x_3$ is $(-x_2, x_1, 0)$ and its flow is a uniform rotation around the axis $(0, 0, 1)$ with unit angular velocity. We will denote by $s = (s_{11}, s_{12}, s_{13}, s_{21}, s_{22}, s_{23})$ the points of $S_{n/2}^2 \times S_{n/2}^2$, that we embed in $\mathbb{R}^3 \times \mathbb{R}^3$, and write indifferently either $s \in \mathbb{R}^6$ or $(s_1, s_2) \in \mathbb{R}^3 \times \mathbb{R}^3$ with $s_i = (s_{i1}, s_{i2}, s_{i3})$. With the 2-form $\sigma_{n/2} + \sigma_{n/2}$, the Hamiltonian vector field H^\sharp of a function $H : S_{n/2}^2 \times S_{n/2}^2 \rightarrow \mathbb{R}$ is thus

$$H^\sharp(s_1, s_2) = - \begin{pmatrix} s_1 \times \text{grad}_{s_1} H(s_1, s_2) \\ s_2 \times \text{grad}_{s_2} H(s_1, s_2) \end{pmatrix}. \quad (6)$$

The integrable part H_0 of the n -shell normal form (5), that we call the Pauli Hamiltonian, is

$$H_0(s) = \omega_1(\beta) s_{13} + \omega_2(\beta) s_{23}$$

where

$$\omega_1(\beta) = \sqrt{1 + \beta}, \quad \omega_2(\beta) = \sqrt{1 - \beta}$$

with

$$\beta = \frac{2\mathcal{B}_n \cdot \mathcal{E}_n}{\|\mathcal{B}_n\|^2 + \|\mathcal{E}_n\|^2}.$$

The parameter β takes values in the closed interval $[-1, 1]$ and depends on n , on the strengths of the fields and on the angle γ between them. The fields are necessarily parallel ($\gamma = 0$) for $\beta = 1$ and antiparallel ($\gamma = \pi$) for $\beta = -1$; the value $\beta = 0$ corresponds to orthogonal fields ($\gamma = \pi/2$), including the Stark ($\mathcal{B} = 0$) and Zeeman ($\mathcal{E} = 0$) limits; for each other value of β , the angle satisfies $0 \leq \gamma < \pi/2$ if $\beta > 0$ and $\pi/2 < \gamma \leq \pi$ if $\beta < 0$.

For each β , the flow of H_0 consists of uniform rotations along the parallels of the two spheres, with angular frequencies $\omega_1(\beta)$ and $\omega_2(\beta)$. This flow, which is sometimes called ‘Pauli precession’, is periodic if $\omega_1(\beta)/\omega_2(\beta) \in \mathbb{Q}$, quasi-periodic otherwise.

Finally, written in Cartesian coordinates in the embedding space $\mathbb{R}^3 \times \mathbb{R}^3$, the function P in (5) is a polynomial in ϵ , of an order that depends on the order of the normalization.

3 Nekhoroshev theory for the n -shell normal form

We now describe our application of Nekhoroshev theory to the n -shell normal form $H_0 + \epsilon P$. In order to do this we need to introduce some objects and notation.

A. Transformation to unit spheres. Even though it is not strictly necessary, we prefer working on unit spheres, rather than on spheres of radius $n/2$.

With the conformal symplectomorphism $s \mapsto \frac{2}{n}s$ and a corresponding reparameterization of time, the n -shell normal form can be regarded as defined by the Hamiltonian (5) on the phase space $S^2 \times S^2$, where now S^2 is the unit sphere in \mathbb{R}^3 , equipped with the symplectic structure $\sigma := \sigma_1 + \sigma_1$. The Hamiltonian vector field H^\sharp of a function $H : S^2 \times S^2 \rightarrow \mathbb{R}$, relative to the symplectic structure σ , is still given by (6). Thus, after the time-reparameterization, the flow of H_0 consists of uniform rotations on the parallels of the two spheres, with frequencies $\omega_1(\beta)$ and $\omega_2(\beta)$.

From now on, we consider the Hamiltonian (5) on $S^2 \times S^2$.

We denote by I_1 and I_2 the restrictions to $S^2 \times S^2$ of the two functions s_{13} and s_{23} and, using a standard terminology in Hamiltonian mechanics, we call them *actions* because their flows are periodic with period 2π . Furthermore, we write $I = (I_1, I_2) : S^2 \times S^2 \rightarrow \mathbb{R}$ and we use the *frequency vector map*

$$\omega : [-1, 1] \rightarrow \mathbb{R}^2, \quad \omega(\beta) = (\omega_1(\beta), \omega_2(\beta)). \quad (7)$$

Correspondingly, we write the Pauli Hamiltonian H_0 as

$$H_0 = \omega(\beta) \cdot I.$$

B. Geography of resonances. Since the parameter β takes values in the closed interval $[-1, 1]$, the set of all possible values of the frequency vector $\omega(\beta) = (\omega_1(\beta), \omega_2(\beta))$ is the closed arc Ω of the circle $\omega_1^2 + \omega_2^2 = 2$ which lies in the first quadrant $\omega_1 \geq 0, \omega_2 \geq 0$, see Figure 1, left. We call *frequency space* the set Ω .

A frequency vector $\omega(\beta) \in \Omega$ is *resonant* with an integer vector $\nu = (\nu_1, \nu_2) \in \mathbb{Z}^2$ if $\omega(\beta) \cdot \nu = 0$; we also say that ν is a *resonance* for the considered value of β (sometimes in the literature it is $\omega(\beta)$ that is called the resonance). The norm $|\nu| := |\nu_1| + |\nu_2|$ is the *order* of the resonance. Since the frequency space Ω spans the entire first quadrant, any integer vector ν such that $\nu_1 \nu_2 \leq 0$ is a resonance for some $\beta \in [-1, 1]$. We may clearly consider only resonances of minimal order, such that $\gcd(\nu_1, \nu_2) = 1$, and need not distinguish between ν and $-\nu$. Thus, the set of possible resonances consists of all vectors in

$$\mathbb{Z}_*^2 = \{(\nu_1, \nu_2) \in \mathbb{Z}^2 \setminus \{(0, 0)\} : \gcd(\nu_1, \nu_2) = 1, \nu_1 \leq 0, \nu_2 \geq 0\}.$$

Nekhoroshev approach depends on a parameter, the cutoff N . For each $N > 1$, this approach considers only resonances of order up to N and decomposes the action space of the system into regions of approximate resonance with these resonances. In the present case, we decompose instead the frequency space. Let

$$\mathbb{Z}_N^2 := \{\nu \in \mathbb{Z}_*^2 : 0 < |\nu| \leq N\}.$$

We define the N -resonant region of $\nu \in \mathbb{Z}_N^2$ as the subset

$$\Omega_{N, \nu} := \left\{ \omega \in \Omega : |\omega \cdot \nu| \leq \frac{1}{\sqrt{2}N} \right\}$$

of the frequency space Ω , see Figure 1, right. Thus, $\Omega_{N, \nu}$ is the intersection of Ω with the angle centered on the line orthogonal to ν and of semi-amplitude θ_ν such that

$$\sin \theta_\nu = \frac{1}{2N \|\nu\|}$$

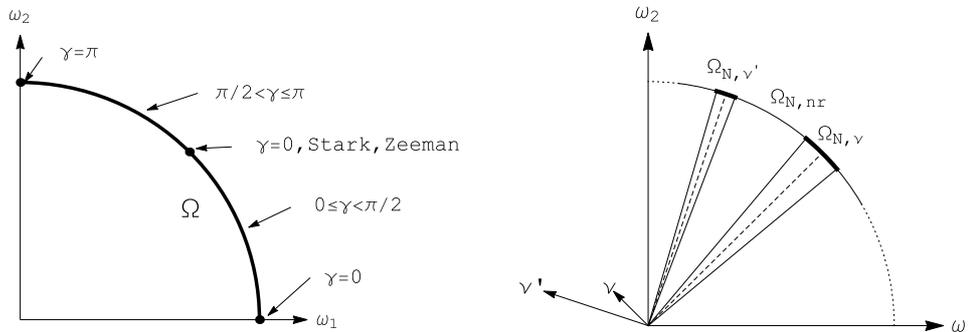


Figure 1: The frequency space Ω (left) and a schematic representation of its decomposition into resonant and nonresonant regions (right).

(recall that $\|\omega\| = \sqrt{2}$). This choice of the N -resonant regions guarantees that any two different of them have empty intersection (the angle $\widehat{\nu\nu'} \in [0, \pi/2)$ between two nonzero nonparallel vectors ν and ν' of \mathbb{R}^2 satisfies $\sin(\widehat{\nu\nu'}) = \frac{|\det(\nu, \nu')|}{\|\nu\|\|\nu'\|} \geq \frac{1}{\|\nu\|\|\nu'\|}$). Therefore, the complement

$$\Omega_{N, nr} := \Omega \setminus \bigcup_{\nu \in \mathbb{Z}_N^2} \Omega_{N, \nu}$$

in Ω of all the resonant regions of order $\leq N$, which is called the N -nonresonant region, is a union of arcs. Between any two resonant regions of order $\leq N$ there is an arc of the N -nonresonant region. From these definitions it follows that it is possible to control the ‘small denominators’ in the resonant and nonresonant regions as follows:

$$\omega \in \Omega_{\bar{\nu}}, \bar{\nu} \in \mathbb{Z}_N^2 \implies |\omega \cdot \nu| \geq \frac{1}{\sqrt{2}N} \quad \forall \nu \in \mathbb{Z}^2 \text{ such that } 0 < |\nu| \leq N, \nu \not\parallel \bar{\nu} \quad (8)$$

$$\omega \in \Omega_{N, nr} \implies |\omega \cdot \nu| \geq \frac{1}{\sqrt{2}N} \quad \forall \nu \in \mathbb{Z}^2 \text{ such that } 0 < |\nu| \leq N. \quad (9)$$

Remarks: (i) The size $\sim N^{-1}\|\nu\|^{-1}$ of the resonant regions is typical of Nekhoroshev’s theory for perturbations of convex Hamiltonians with 2 degrees of freedom, see e.g. [27, 3].

(ii) Reference [10] uses a three-dimensional parameter space $C \subset \mathbb{R}^3 \ni (s, a^2, d)$, with $s = \epsilon/n$, $a^2 = \|\mathcal{B}_n\|^2/\epsilon^2$ and $d = \beta/2$. Since (a^2, d) belong to the disk of radius $1/2$ centered at $(1/2, 0)$ in \mathbb{R}^2 , the set C is an infinite solid cylinder, cf. Figure 3 of [10]. Our frequency space Ω can be thought of as the image of C under the map $(s, a^2, d) \mapsto \omega(2d)$ where ω is the frequency vector map (7). Therefore, the level sets of the frequency vector in the space $\mathbb{R}^3 \ni (s, a^2, d)$ are the planes $d = \text{const}$. The pullback to C of our (non)resonant regions in Ω consist of slices of C parallel to the planes $d = \text{const}$. Their intersection with the disks $s = \text{const}$ should be compared to the *resonant zones* in Figure 3 of [10].

(iii) Many field orientations and relative strengths can result in the same resonance. This applies particularly to collinear fields, which can be tuned to *any* resonance. On the other hand, the $1:1$ and $1:0$ (or $0:1$) resonances occur only for the orthogonal and parallel (or antiparallel) field configurations, respectively. So our parameterization in this article, while sufficient for the Nekhoroshev theory, does not reflect on the mutual orientation of the fields, which requires an additional parameter (such as a^2 in [10]).

C. A symplectic atlas and Fourier series on the sphere. In Nekhoroshev theory, normal forms need not be actually computed, but it is necessary to prove that they exist up to a sufficiently high order, and to provide (recurrent) estimates on them. The way of doing this which is most used is through the introduction of action-angle coordinates, that makes possible to resort to standard techniques from Fourier analysis and analytic functions. In our case, however, for the reasons mentioned in the Introduction, we need to do this construction globally on the phase space $S^2 \times S^2$, and this requires special considerations. In order to do this, we will make use of a symplectic atlas for $S^2 \times S^2$ made of several charts. We first discuss this topic for a single sphere S^2 .

Let σ_1 be the area form on S^2 . We consider a symplectic atlas for (S^2, σ_1) made of the following three charts. The first chart covers the sphere minus the North and South poles $(0, 0, \pm 1)$ and is formed by a pair of cylindrical-like action-angle coordinates $(j, \varphi) \in (-1, 1) \times S^1$ such that

$$x_1 = \sqrt{1 - j^2} \cos \varphi, \quad x_2 = \sqrt{1 - j^2} \sin \varphi, \quad x_3 = j. \quad (10)$$

Clearly $\sigma_1 = d\varphi \wedge dj$. The coordinate j coincides with the (restriction to this chart domain of the) ‘action’ x_3 and the local representative of X_{x_3} is $\frac{\partial}{\partial \varphi}$. The second chart has for domain the sphere minus the South Pole and Cartesian-like coordinates (p_+, q_+) which take values in the disk $p_+^2 + q_+^2 < 4$ of \mathbb{R}^2 , such that

$$x_1 = -p_+ \sqrt{1 - \frac{p_+^2 + q_+^2}{4}}, \quad x_2 = q_+ \sqrt{1 - \frac{p_+^2 + q_+^2}{4}}, \quad x_3 = 1 - \frac{p_+^2 + q_+^2}{2}. \quad (11)$$

Now $\sigma_1 = dq_+ \wedge dp_+$ and the local representative of X_{x_3} is $p_+ \frac{\partial}{\partial q_+} - q_+ \frac{\partial}{\partial p_+}$. The third chart, which excludes the North Pole, is constructed similarly to the second one, just with some signs reversed:

$$x_1 = p_- \sqrt{1 - \frac{p_-^2 + q_-^2}{4}}, \quad x_2 = q_- \sqrt{1 - \frac{p_-^2 + q_-^2}{4}}, \quad x_3 = -1 + \frac{p_-^2 + q_-^2}{2}. \quad (12)$$

The second and third charts (that, in cartography are known as Lambert azimuthal projections, centered on the North and South Poles respectively) provide an atlas for S^2 . However, later on we will restrict the domain of each of these charts to a neighbourhood of the Pole, in which the Taylor series of the local representative of the Hamiltonian (5) is convergent. This is why we consider the action-angle chart as well.

Since the transition functions between the three considered charts are real analytic, this atlas gives S^2 the structure of a real analytic manifold.

We say that a function $F : S^2 \rightarrow \mathbb{R}$ is a *real analytic function on S^2* if the transition functions between its local representatives in these three charts are real analytic.³ From expressions (10), (11) and (12) it is clear that if a function $\tilde{F} : V \rightarrow \mathbb{R}$ from a neighbourhood V of S^2 in \mathbb{R}^3 is real analytic, then its restriction to S^2 is a real analytic function on S^2 .

At the heart of perturbation theory is the possibility of solving the so called ‘homological equation’, that in our case—and adapted to the case of a single sphere—involves the linear differential operator

$$F \mapsto \{x_3, F\}$$

on the algebra of (real analytic) functions on S^2 . Solving this equation requires a Fourier series for functions on S^2 . Written in coordinates, this should of course coincide with the standard Fourier

³Even though this is not strictly necessary for the sequel, we note that the class of real analytic functions defined in this way is independent of the chosen atlas. In fact, by the Grauert-Morrey theorem [18, 22], any smooth manifold has a unique real analytic structure compatible with its differentiable structure. Therefore, the transition functions between any pair of charts that belong to any pair of real analytic atlases of S^2 is real analytic. It follows that a function which is real analytic according to our definition has real analytic representatives in any chart of any real analytic atlas of S^2 .

series in the action-angle chart; the appropriate extension to neighbourhoods of the poles is similar to the Fourier expansion used in the neighbourhood of an elliptic point in \mathbb{R}^n , introduced in [4, 13].

Proposition 1. *Any real analytic function $F : S^2 \rightarrow \mathbb{R}$ can be expanded into the absolutely convergent series*

$$F = \sum_{\nu \in \mathbb{Z}} F_\nu \quad (13)$$

where, for each $\nu \in \mathbb{Z}$, $F_\nu : S^2 \rightarrow \mathbb{R}$ is a real analytic function that satisfies

$$\{x_3, F_\nu\} = i\nu F_\nu. \quad (14)$$

The local representatives of F_ν in the three coordinate systems are given in the proof.

Proof. We first define the local representatives of the functions F_ν in the three charts of the considered atlas and then verify that they match in the intersections of the domains.

The local representative $f(j, \varphi)$ of F in the action-angle chart can be expanded in the standard Fourier series

$$f(j, \varphi) = \sum_{\nu \in \mathbb{Z}} \hat{f}_\nu(j) e^{i\nu\varphi} \quad (15)$$

which is absolutely convergent in the entire coordinate domain.

Introduce now the symplectic complex coordinates

$$z = \frac{p_+ + iq_+}{\sqrt{2}}, \quad w = \frac{p_+ - iq_+}{\sqrt{2}i}$$

in the domain of the northern polar chart (11); they take value in the open set $Q_2 := \{(z, w) \in \mathbb{C}^2 : \bar{w} = iz, |z|^2 < 2\}$. The local representative $f^+(z, w)$ of F in these coordinates can be expanded in Taylor series centered at $(0, 0)$,

$$f^+(z, w) = \sum_{m, n=0}^{\infty} f_{m, n}^+ z^m w^n$$

which is absolutely convergent in an open subset Q_{r_+} of Q_2 , with $Q_{r_+} = \{(z, w) \in Q_2 : |z|^2 < r_+\}$ for some $r_+ > 0$ that depends on the analyticity properties of f^+ . The radius r_+ is not unique, so we make a choice for it. The Taylor series of f^+ can be rewritten as

$$\sum_{\nu=-\infty}^{-1} \tilde{f}_\nu^+(izw) w^{|\nu|} + \sum_{\nu=0}^{\infty} \tilde{f}_\nu^+(izw) z^\nu$$

where, for each $y \geq 0$,

$$\tilde{f}_\nu^+(y) = \sum_{m=0}^{\infty} f_{m+|\nu|, m}^+ (-iy)^m \quad \text{if } \nu < 0 \quad (16)$$

while

$$\tilde{f}_\nu^+(y) = \sum_{m=0}^{\infty} f_{m, m+\nu}^+ (-iy)^m \quad \text{if } \nu \geq 0, \quad (17)$$

and all these series are absolutely convergent in Q_{r_+} . Thus, if for each $\nu \in \mathbb{Z}$ we define

$$E_\nu(z, w) = w^\nu \quad \text{if } \nu \geq 0, \quad E_\nu(z, w) = z^{|\nu|} \quad \text{if } \nu < 0,$$

we have

$$f^+(z, w) = \sum_{\nu \in \mathbb{Z}} \tilde{f}_\nu^+(izw) E_\nu(z, w) \quad \forall (z, w) \in Q_{r_+}, \quad (18)$$

with an absolutely convergent series.

We introduce in a similar way complex coordinates in the domain of the southern polar chart. The Taylor series of f^- centered at $(0, 0)$ converges in a nonempty domain Q_{r_-} , with some $r_- > 0$, and we arrive at the series representation

$$f^-(z, w) = \sum_{\nu \in \mathbb{Z}} \tilde{f}_\nu^-(izw) E_\nu(z, w) \quad \forall (z, w) \in Q_{r_-} \quad (19)$$

which is absolutely convergent in Q_{r_-} .

The preimage in S^2 of the restricted domains Q_{r_\pm} of the two complex polar charts are the two spherical caps $\pm x_3 > 1 - r_\pm$. Therefore, together with the action-angle chart, the restrictions of the two complex polar charts to the domains Q_{r_+} and Q_{r_-} still form an atlas for S^2 . It is now easy to verify that, for each $\nu \in \mathbb{Z}$, the functions $f_\nu e^{i\nu\varphi}$, $\tilde{f}_\nu^+ E_\nu$ and $\tilde{f}_\nu^- E_\nu$ match in the intersections of the chart domains of this atlas, and are thus the local representatives of a function $F_\nu : S^2 \rightarrow \mathbb{R}$. In fact, in the intersection of the domain of the action-angle chart and of the northern polar chart it is $p_+ = -\sqrt{2}\sqrt{1-j} \cos \varphi$, $q_+ = \sqrt{2}\sqrt{1-j} \sin \varphi$ and hence

$$w = i\sqrt{1-j} e^{i\varphi}, \quad z = -\sqrt{1-j} e^{-i\varphi}, \quad j = 1 - izw.$$

Thus, equating the series expansions (15) and (18) of the local representatives of F in these two charts gives

$$\hat{f}_\nu e^{i\nu\varphi} = \tilde{f}_\nu^+ E_\nu \quad \forall \nu \in \mathbb{Z}$$

in the spherical segment $x_3 > 1 - r_+$ of S^2 . Similarly

$$\hat{f}_\nu e^{i\nu\varphi} = \tilde{f}_\nu^- E_\nu \quad \forall \nu \in \mathbb{Z}$$

in the spherical segment $x_3 < -(1 - r_-)$ of S^2 . Since the intersection of the preimages in S^2 of the restricted domains Q_{r_\pm} of the polar charts is contained within the preimage of the domain of the action-angle chart, it follows from here that the two functions \tilde{f}_ν^\pm match.⁴ Furthermore, this fact implies that the function F_ν is independent of the choice of r_\pm , as long as they are positive.

This proves, for each ν , the existence of a function F_ν on S^2 whose local representatives in the three charts are $f_\nu e^{i\nu\varphi}$, $\tilde{f}_\nu^+ E_\nu$ and $\tilde{f}_\nu^- E_\nu$. F_ν is real analytic because its local representatives are real analytic, being given by absolutely convergent power series. The absolute convergence of the series (13) follows from that of the local representatives. ■

The function F_ν will be called the ν -th harmonic of F and (13) will be called the *Fourier series* of F .

Remarks (i) The local representative in the complex northern polar chart of the function F_ν is defined in the entire domain Q_2 of these coordinates even if the *maximum* convergence radius r_+ of the Taylor series of f^+ is < 2 . In this case the local representative of F_ν may still be written as $\tilde{f}_\nu^+ E_\nu$, given that E_ν is nonzero outside Q_{r_+} , but the function \tilde{f}_ν^+ has the series representation (16), (17) only in Q_{r_+} .

(ii) The function $\hat{f}_\nu(j) e^{i\nu\varphi}$ is the local representative of a function defined on the entire sphere; the two functions $\hat{f}_\nu(j)$ and $e^{i\nu\varphi}$, separately, are not.

All this extends in an obvious way to the phase space $S^2 \times S^2$, that can be covered with a symplectic atlas made of nine charts, given by all possible combinations of the three charts on each sphere. Any smooth function $F : S^2 \times S^2 \rightarrow \mathbb{R}$ can be expanded in the Fourier series $\sum_{\nu \in \mathbb{Z}^2} F_\nu$ of its harmonics F_ν , now with $\nu = (\nu_1, \nu_2) \in \mathbb{Z}^2$. For instance, the local representatives f of F and f_ν

⁴In practice, the convergence radii r_\pm of f^\pm may quite easily be smaller than 1. In that case, after the restrictions to Q_{r_\pm} , the two complex polar charts do not intersect at all. In Section 4 we will *de facto* assume that $r_\pm < 1$.

of F_ν in a chart with action-angle coordinates (j_1, φ_1) on the first sphere and complex coordinates (z_2, w_2) on the second satisfy

$$\begin{aligned} f(j_1, \varphi_1, z_2, w_2) &= \sum_{\nu_1 \in \mathbb{Z}} \sum_{m, n \in \mathbb{N}} \tilde{f}_{\nu_1, m, n}(j_1) e^{i\nu_1 \varphi_1} z_2^m w_2^n \\ &= \sum_{\nu \in \mathbb{Z}^2} \hat{f}_\nu(j_1, i z_2 w_2) e^{i\nu_1 \varphi_1} E_{\nu_2}(z_2, w_2) \\ &= \sum_{\nu \in \mathbb{Z}^2} f_\nu(j_1, \varphi_1, z_2, w_2). \end{aligned} \quad (20)$$

For any given $\omega = (\omega_1, \omega_2) \in \mathbb{R}^2$, the harmonics of $F : S^2 \times S^2 \rightarrow \mathbb{R}$ satisfy

$$\{\omega \cdot I, F_\nu\} = i\omega \cdot \nu F_\nu$$

where as above $I = (I_1, I_2)$. A *resonant harmonic* of $F : S^2 \times S^2 \rightarrow \mathbb{R}$ is a harmonic F_ν such that ν is a resonance for $\omega(\beta)$. For either $\bar{\nu} \in \mathbb{Z}_*^2$ a vector of minimal order or $\bar{\nu} = (0, 0)$ define the projector $\Pi_{\bar{\nu}}$ on the space of functions on $S^2 \times S^2$ as

$$\Pi_{\bar{\nu}} F = \sum_{\nu \in \mathbb{Z}\bar{\nu}} F_\nu$$

where the sum is restricted to all $\nu \in \mathbb{Z}^2$ parallel to $\bar{\nu}$, that is, $\nu = k\bar{\nu}$ for some $k \in \mathbb{Z}$. Clearly, $\Pi_0 F = F_0$ is a function of the actions s_{13}, s_{23} alone and equals the average of F over the orbits of the $S^1 \times S^1$ -action given by uniform rotations along the parallels of the two spheres. For any $\bar{\nu} \neq 0$, $\Pi_{\bar{\nu}} F$ is the average of F over the orbits of the S^1 -action given by uniform rotations along the parallels of the two spheres with any frequency $\omega = (\omega_1, \omega_2) \neq 0$ which is resonant with $\bar{\nu}$.

D. Nekhoroshev theorem for the n -shell normal form. We may now state a version of Nekhoroshev theorem for the n -shell normal form Hamiltonian $H_0 + \epsilon P$:

Proposition 2. *There exist positive constants ϵ_* and c , independent of ϵ , such that the following is true for any $0 < \epsilon < \epsilon_*$ and any*

$$N \leq c \sqrt{\epsilon_* / \epsilon}.$$

For each $\beta \in [-1, 1]$ there exists a real analytic symplectic diffeomorphism Ψ of $S^2 \times S^2$ onto itself which satisfies

$$\Psi = \text{id} + \mathcal{O}(\sqrt{\epsilon / \epsilon_*})$$

and conjugates $H_0 + \epsilon P$ to

$$H_0 + \epsilon G + \epsilon e^{-N/c} F$$

with functions $G, F : S^2 \times S^2 \rightarrow \mathbb{R}$ which are bounded uniformly in ϵ together with their first derivatives, and

- i. if $\omega(\beta) \in \Omega_{N, \text{nr}}$ then $G = \Pi_0 G = \Pi_0 P + \mathcal{O}(\sqrt{\epsilon / \epsilon_*})$*
- ii. if $\omega(\beta) \in \Omega_{N, \nu}$ for some (minimal order) resonance ν of order $\leq N$, then $G = \Pi_\nu G = \Pi_\nu P + \mathcal{O}(\sqrt{\epsilon / \epsilon_*})$.*

The proof is deferred to the next section.

Within the standard formulation of Nekhoroshev theorem, the cutoff N is chosen in such a way to minimize the remainder for each value of ϵ , that is, $N = c\sqrt{\epsilon_* / \epsilon}$, because this leads to the longest stability times. Time, however, does not enter our problem. Therefore, we prefer leaving the cutoff unevaluated.

Proposition 2 answers the question raised at the beginning: if resonances up to order N are taken into consideration, then the size of the ν -resonant zones decays with N and with the order of the resonance as in (3), that is, as

$$\frac{1}{N\|\nu\|}.$$

In the adopted construction, the size of each arc of the resonant region depends on the orders of the two nearby resonances of order up to N , and is not less than a quantity of the order of N^{-2} . If the cutoff is linked to the perturbation parameter as in Nekhoroshev theory, then the size of the resonant regions is $\sim \sqrt{\epsilon}/\|\nu\|$ while that of the nonresonant arcs is bounded from below by $\sim \epsilon$.

From a dynamical point of view, Proposition 2 implies that, on a time scale of the order of $e^{N/c}$, the first integrals of the truncated normal form $H_0 + \epsilon G$ are approximate first integrals of the n -shell normal form Hamiltonian $H_0 + \epsilon P$. Specifically, if $J : S^2 \times S^2 \rightarrow \mathbb{R}$ is a function such that $\{H_0 + \epsilon G, J\} = 0$, then along the flow of $H_0 + \epsilon P$ the function $J \circ \Psi$ varies at most of quantities of order $\sqrt{\epsilon/\epsilon_*}$ on times up to $\sim e^{N/c}$. In the nonresonant case, G does not contain nonzero harmonics and the two actions I_1 and I_2 are integrals of motion of $H_0 + \epsilon G$. In the resonant case, instead, the linear combination $\nu \cdot I = \nu_1 I_1 + \nu_2 I_2$ is a first integral of $H_0 + \epsilon G$. In all cases, the truncated normal form Hamiltonian $H_0 + \epsilon G$ has two first integrals in involution, and is integrable.

4 Proof of Proposition 2

The proof of Proposition 2 is rather standard, except for the fact that the phase space $S^2 \times S^2$ cannot be covered with a single coordinate system. The normal forms can be constructed globally on $S^2 \times S^2$ using the Fourier series of Proposition 1, but in order to produce estimates on these normal forms we need to resort to an atlas made of different charts.

Basically, all the estimates we need can be found elsewhere: for instance, those using action-angle coordinates in [3], those using complex coordinates in [13], and those using both systems of coordinates in [4]. Thus, we will not reprove all of them here. For clarity, however, we will express these estimates as estimates on functions on $S^2 \times S^2$, rather than on their local representatives. This will require some preliminary work, but has the advantage of making the perturbation construction more transparent.

A. Complex domains, function classes and norms. We will use an atlas for $S^2 \times S^2$ made of the nine products of a system of action-angle coordinates (j, φ) and two complex coordinate systems (z, w) on each sphere; the complex coordinates have been introduced in the proof of Proposition 1 and we will refer to them as to the ‘North’ and ‘South’ coordinates. As is typical of perturbation theory we need complex extensions of these nine chart domains.

The construction of the complex domains depends on two positive numbers R_z and R_φ , that will be chosen later based on the analyticity properties of the perturbation P ; for now, they may be considered as parameters. We assume that

$$R_z \leq 1, \quad R_\varphi \leq 1;$$

this condition will be tacitly used at several points in the sequel.

We start from an atlas on each sphere formed by the restrictions of the North and South coordinate domains to

$$\{(z, w) \in \mathbb{C} : \bar{w} = iz, |z| < R_z\}$$

and by the restriction of the action-angle coordinates to

$$\{(j, \varphi) \in (-1, 1) \times S^1 : |j| < 1 - R_z^2/4\}$$

(since $izw = 1 - j$, with these choices the intersection of any pair of adjacent coordinate domains is a nonempty spherical segment).

For any $r > 0$, denote by $B_r(x) \subset \mathbb{C}$ the complex disk of radius r and center $x \in \mathbb{R}$ and consider the complex neighbourhood of the torus

$$S_r^1 = \{\varphi \in \mathbb{C}/(2\pi\mathbb{Z}) : \text{Im}(\varphi) < r\}.$$

Write $R = (R_z, R_\varphi)$. For any $\rho = (\rho_z, \rho_\varphi)$ such that $0 < \rho \leq R$ (inequalities among vectors are interpreted componentwise) consider the complex two-disk

$$\Delta_\rho = B_{\rho_z}(0) \times B_{\rho_\varphi}(0) \ni (z, w),$$

that extends (a subset of) the domain of the North and South coordinates to the complex, and the complex set

$$K_\rho = J_{\rho_z} \times S_{\rho_\varphi}^1 \ni (j, \varphi) \quad \text{with} \quad J_{\rho_z} = \bigcup_{j \in \mathbb{R}, |j| < 1 - R_z^2/4} B_{\rho_z^2/4}(j),$$

that extends (a subset of) the domain of the action-angle coordinates. If $R/2 \leq \rho \leq R$ then the preimages of the real parts of the three complex domains Δ_ρ (North), K_ρ and Δ_ρ (South) cover the entire sphere.

Thus, taking all possible nine products of the sets Δ_ρ and K_ρ on each sphere gives complex domains

$$D_\rho^\alpha, \quad \alpha = 1, \dots, 9,$$

and the preimages under the coordinate maps of their real parts cover entirely $S^2 \times S^2$ if $R/2 \leq \rho \leq R$. We will denote by f^α , $\alpha = 1, \dots, 9$, the local representatives in these charts of a function $F : S^2 \times S^2 \rightarrow \mathbb{R}$.

We say that a real function on $S^2 \times S^2$ is of class \mathcal{A}_ρ if it is real analytic and its local representatives in all nine charts have complex analytic extensions to the sets D_ρ^α , bounded together with their first derivatives.

Any function $F \in \mathcal{A}_\rho$ can be expanded in its Fourier series $\sum_{\nu \in \mathbb{Z}^2} F_\nu$, and its harmonics $F_\nu \in \mathcal{A}_\rho$. We may thus define a norm in \mathcal{A}_ρ out of the usual Fourier-like norms of its local representatives; the use of such a norm in action-angle coordinates is completely standard in perturbation theory, while for the case of complex coordinates see [4, 13]. Specifically, for any $F \in \mathcal{A}_\rho$ we define

$$|F|_\rho := \sum_{\nu \in \mathbb{Z}^2} |F_\nu|_\rho^\infty$$

where, for each $\nu \in \mathbb{Z}^2$,

$$|F_\nu|_\rho^\infty := \max_{\alpha=1, \dots, 9} \sup_{y \in D_\rho^\alpha} |f_\nu^\alpha(y)|.$$

Lemma 1. *Assume $F \in \mathcal{A}_\rho$ for $R/2 < \rho \leq R$. Then for any $\nu \in \mathbb{Z}^2$ and any $x > 0$ such that $xR < \rho/2$,*

$$|F_\nu|_{\rho-xR}^\infty \leq e^{-x|\nu|R_\varphi} |F_\nu|_\rho^\infty.$$

Proof. Let us for instance verify this inequality for the representatives in the chart with coordinates $(j_1, \varphi_1, z_2, w_2)$, see (20). Since the maximum of the modulus of an analytic function is reached at the boundary of its domain,

$$\sup_{K_\delta \times \Delta_\delta} |f_\nu(j_1, \varphi_1, z_2, w_2)| = e^{|\nu_1|\delta_\varphi} \delta_z^{|\nu_2|} \sup_{K_\delta \times \Delta_\delta} |\hat{f}_\nu(j_1, iz_2 w_2)|$$

for any $\delta = (\delta_z, \delta_\varphi)$. Therefore, writing for shortness $|\hat{f}_\nu|_\delta^\infty$ for $\sup_{K_\delta \times \Delta_\delta} |\hat{f}_\nu(j_1, iz_2 w_2)|$,

$$\begin{aligned} |f_\nu|_{\rho-xR}^\infty &= e^{|\nu_1|(\rho_\varphi - xR_\varphi)} (\rho_z - x\rho_z)^{|\nu_2|} |\hat{f}_\nu|_{\rho-xR}^\infty \\ &\leq e^{-x|\nu_1|R_\varphi} (1-x)^{|\nu_2|} |\hat{f}_\nu|_\rho^\infty \\ &\leq e^{-x|\nu_1|R_\varphi} e^{-x|\nu_2|} |\hat{f}_\nu|_\rho^\infty \\ &\leq e^{-x(|\nu_1|+|\nu_2|)R_\varphi} |\hat{f}_\nu|_\rho^\infty \end{aligned}$$

where we have used $\rho_z < 1$ and $1-x < e^{-x}$ and, in the last inequality, $R_\varphi < 1$. The computations in the other charts are analogous. ■

Lemma 1 has the following consequence. Given a ‘cutoff’ $N > 0$, define the ‘ultraviolet’ part of a function $F : S^2 \times S^2 \rightarrow \mathbb{R}$ as $F^{>N} = \sum_{|\nu|>N} F_\nu$, and write $F = F^{\leq N} + F^{>N}$. Then, if $F \in \mathcal{A}_\rho$ and $xR < \rho/2$,

$$|F^{>N}|_{\rho-xR} \leq e^{-NxR_\varphi} |F^{>N}|_\rho. \quad (21)$$

We need also a norm for the Hamiltonian vector fields of functions. The Hamiltonian vector field F^\sharp of a function $F \in \mathcal{A}_\rho$ is defined by $\sigma(F^\sharp, \cdot) = dF$. By linearity,

$$F^\sharp = \sum_{\nu \in \mathbb{Z}^2} F_\nu^\sharp$$

where, for each ν , F_ν^\sharp is the Hamiltonian vector field of the ν -th harmonic F_ν of F . We thus define

$$\|F^\sharp\|_\rho := \sum_{\nu \in \mathbb{Z}^2} \|F_\nu^\sharp\|_\rho^\infty$$

where, for each ν ,

$$\|F_\nu^\sharp\|_\rho^\infty := \max_{\alpha=1,\dots,9} \max_{\xi} \sup_{y \in D_\rho^\alpha} \frac{1}{R_\xi} |(f_\nu^\alpha)_\xi^\sharp(y)|$$

where ξ runs over all the coordinates in the chart α , $(f_\nu^\alpha)_\xi^\sharp$ is the ξ -component of the Hamiltonian vector field of ν -th harmonic f_ν^α of the local representative f^α of F in the chart α , and, in charts with action-angle coordinates, $R_j = R_z^2$. Similarly to (21),

$$\|(F^{>N})^\sharp\|_{\rho-xR} \leq e^{-NxR_\varphi} \|(F^{>N})^\sharp\|_\rho. \quad (22)$$

B. The Lie method. The next tool we need is the so called ‘Lie method’ to construct near the identity symplectic diffeomorphisms (we use the version of this method for vector fields described in [11]). This method is particularly convenient in the present case because it constructs the diffeomorphisms as time-one maps of Hamiltonian flows and therefore works seamlessly on manifolds.

Given a function $X \in \mathcal{A}_\rho$, denote by L_X the Poisson bracket operator $\{X, \cdot\}$ and by Φ_1^X the time-one map of the flow of its Hamiltonian vector field X^\sharp . For any function $Y \in \mathcal{A}_\rho$, write

$$Y \circ \Phi_1^X = Y + R_1^X(Y) = Y + L_X Y + R_2^X(Y).$$

Since $t \mapsto Y \circ \Phi_t^X$ is analytic and $\frac{d}{dt}(Y \circ \Phi_t^X) = (L_X Y) \circ \Phi_t^X$, the remainders R_j^X , $j = 1, 2$, in these expressions have the Taylor series representation $R_j^X(Y) = \sum_{k=j}^\infty \frac{1}{k!} L_X^k Y$.

Estimates on the method can be obtained from here, using the following basic estimates: if $X, Y \in \mathcal{A}_\rho$ and $x > 0$ is such that $\rho - xR > R/2$, then $L_X Y \in \mathcal{A}_{\rho-xR}$ and

$$|L_X Y|_{\rho-xR} \leq \frac{1}{x} \|X^\sharp\|_{\rho-xR} \|Y\|_\rho, \quad \|(L_X Y)^\sharp\|_{\rho-xR} \leq \frac{4}{x} \|X^\sharp\|_\rho \|Y^\sharp\|_\rho.$$

The proof is obtained by estimating the local representatives in each chart separately, what is more or less routine (see e.g. [3]; however, some care has to be taken when using complex coordinates, see [4]).

From this it follows in a relatively easy way that, if $X \in \mathcal{A}_\rho$ and $x > 0$ is such that $\rho - xR > R/2$ and

$$\|X^\sharp\|_\rho \leq \frac{x}{8}, \quad (23)$$

then the local representatives of Φ_1^X map the extended chart domains $D_{\rho-xR}^\alpha$ into D_ρ^α , each one diffeomorphically onto its image, and Φ_1^X is a diffeomorphism of $S^2 \times S^2$ onto itself that satisfies $\Phi_1^X = \text{id} + \mathcal{O}(\|X^\sharp\|_\rho)$. Moreover, if $Y \in \mathcal{A}_\rho$ then $Y \circ \Phi_1^X \in \mathcal{A}_{\rho-xR}$ and estimates on the remainders $R_j^X(Y)$ are easily obtained; for instance, in the sequel we will use the following estimates:

$$|R_1^X(Y)|_{\rho-xR} \leq 2|L_X Y|_{\rho-\frac{x}{2}R}, \quad |R_2^X(Y)|_{\rho-xR} \leq |L_X(L_X Y)|_{\rho-\frac{x}{2}R}, \quad (24)$$

$$\|R_1^X(Y)^\sharp\|_{\rho-xR} \leq \frac{4}{x}\|X^\sharp\|_\rho\|Y^\sharp\|_\rho, \quad \|R_2^X(Y)^\sharp\|_{\rho-xR} \leq \frac{2}{x}\|X^\sharp\|_\rho\|(L_X Y)^\sharp\|_\rho. \quad (25)$$

C. The single perturbation step. We construct the diffeomorphism Ψ of Proposition 2 as the composition of a number of diffeomorphisms, each putting the Hamiltonian in a normal form one order higher than the previous one. Here we describe the single step.

The construction depends on various parameters, whose values will be chosen later; among them, $N > 0$ and $0 < R = (R_z, R_\varphi) \leq (1, 1)$. In order to treat the two cases i. and ii. at once we formally denote by $\bar{\nu} = 0$ the nonresonant case and write Ω_0 for $\Omega_{N, \text{nr}}$. For $\bar{\nu} \neq 0$, we write $\Omega_{\bar{\nu}}$ for $\Omega_{N, \nu}$.

Assume that $\omega(\beta) \in \Omega_{\bar{\nu}}$ for some $0 \leq \|\bar{\nu}\| \leq N$ (as just said, $\bar{\nu} = 0$ means nonresonance). Fix a ρ such that $R/2 < \rho < R$. Consider a function

$$H = H_0 + U + V \in \mathcal{A}_\rho \quad (26)$$

with $\Pi_{\bar{\nu}}U = U$. By (14) of Proposition 1, the ‘homological equation’

$$L_X H_0 = \Pi_{\bar{\nu}}V^{\leq N} - V^{\leq N}$$

has the solution $X = \sum \frac{V_\nu}{i\nu \cdot \omega(\beta)}$, where the (finite) sum is restricted to those $\nu \in \mathbb{Z}^2$ which have norm $0 < |\nu| \leq N$ and are not parallel to $\bar{\nu}$ (that is, $\nu \neq k\bar{\nu}$ with $k \in \mathbb{Z}$). By (8) and (9) and the definitions of the norms

$$|X|_\rho \leq 2N|V^{\leq N}|_\rho, \quad \|X^\sharp\|_\rho \leq 2N\|V^\sharp\|_\rho \quad (27)$$

and $X \in \mathcal{A}_\rho$ (in these estimates we have for simplicity worsened a factor $\sqrt{2}$ to 2). Therefore,

$$\begin{aligned} (H_0 + U + V) \circ \Phi_1^X &= H_0 + L_X H_0 + R_2^X(H_0) + U + R_1^X(U) + V^{\leq N} + V^{>N} + R_1^X(V) \\ &= H_0 + U' + V' \end{aligned}$$

with

$$U' = U + \Pi_{\bar{\nu}}V^{\leq N}, \quad V' = V^{>N} + R_1^X(U + V) + R_2^X(H_0). \quad (28)$$

Clearly $U' \in \mathcal{A}_\rho$ and $\Pi_{\bar{\nu}}U' = U'$.

In order to satisfy condition (23) we need to make the hypothesis that

$$\|V^\sharp\|_\rho \leq \frac{x}{16N}. \quad (29)$$

This ensures that Φ_1^X is a diffeomorphism of $S^2 \times S^2$ onto itself, that $\mathcal{O}(x/N)$ is near the identity, and that $V' \in \mathcal{A}_{\rho-xR}$. Therefore, Φ_1^X conjugates Hamiltonian (26) to

$$H_0 + U' + V' \in \mathcal{A}_{\rho-xR}$$

with U' and V' as in (28).

In view of the iteration of this procedure we need estimates on the Hamiltonian vector field of V' . Observing that $\|(L_X H_0)^\sharp\|_\rho \leq \|(\Pi_{\bar{v}} V^{\leq N} - V^{\leq N})^\sharp\|_\rho \leq \|(V^{\leq N})^\sharp\|_\rho \leq \|V^\sharp\|_\rho$ and using (27), the estimates (25) give

$$\begin{aligned} \|R_1^X(U + V)^\sharp|_{\rho-xR} &\leq \frac{4}{x} \|X^\sharp\|_\rho \|U^\sharp + V^\sharp\|_\rho \leq \frac{8N}{x} (\|U^\sharp\|_\rho + \|V^\sharp\|_\rho) \|V^{\leq N}\|_\rho \\ \|R_2^X(H_0)^\sharp|_{\rho-xR} &\leq \frac{2}{x} \|X^\sharp\|_\rho \|(L_X H_0)^\sharp\|_\rho \leq \frac{8N}{x} \|V^\sharp\|_\rho \|(V^{\leq N})^\sharp\|_\rho \end{aligned}$$

so that, using also (22) to bound $\|(V^{>N})^\sharp\|_{\rho-xR}$,

$$\|(V')^\sharp\|_{\rho-xR} \leq \beta \|V^\sharp\|_\rho, \quad \beta = \max\left(e^{-NxR_\varphi}, \frac{8N}{x} (\|U^\sharp\|_\rho + 2\|V^\sharp\|_\rho)\right). \quad (30)$$

Similar computations give

$$|V'|_{\rho-xR} \leq \beta |V|_\rho. \quad (31)$$

Moreover, up to constants,

$$\Phi_1^X - \text{id} = \mathcal{O}(\|X^\sharp\|_\rho) = \mathcal{O}(N\|V^\sharp\|_\rho). \quad (32)$$

D. The iteration. Since the Hamiltonian $H_0 + \epsilon P$ is a polynomial in $\mathbb{R}^3 \times \mathbb{R}^3$, its restriction to $S^2 \times S^2$ is of class \mathcal{A}_R for some $R = (R_z, R_\varphi) > (0, 0)$. We assume $R_z < 1$ and $R_\varphi < 1$.

Define $\epsilon_* = \frac{R_\varphi}{2^s e \|\bar{P}^\sharp\|_R}$ and choose $N \leq c\sqrt{\epsilon_*}/\epsilon$, with $c = \frac{4}{R_\varphi}$. Iterate the normal form construction described in subsection 4.C. a number $r = \lfloor \frac{NR_\varphi}{2} \rfloor$ of times, each time with $x = \frac{1}{2r}$, starting with the Hamiltonian $H^{(0)} = H_0 + U_0 + V_0$ with $U_0 = 0$ and $V_0 = \epsilon P$. The first time this is possible because, with the above choices of ϵ_* , N and x , $\|V_0^\sharp\|_{\rho_0} = \epsilon \|P^\sharp\|_R = \frac{R_\varphi}{2^s e} \frac{\epsilon}{\epsilon_*} = \frac{1}{16eR_\varphi N^2} \leq \frac{x}{16eN}$ and (29) is satisfied.

Assume now by induction that the procedure can be performed s times, for some $s \leq r$, producing the Hamiltonians $H^{(1)}, \dots, H^{(s)}$, with

$$H^{(j)} = H_0 + U_j + V_j \in \mathcal{A}_{\rho_j}$$

with $U_j = \sum_{h=0}^{j-1} \Pi_{\bar{v}} V_h^{\leq N}$ and $\|V_j^\sharp\|_{\rho_j} \leq e^{-j} \epsilon \|P\|_R$ for all $j = 1, \dots, s$, where $\rho_j = R - jxR = R - \frac{j}{2r}R$.

Under this hypothesis, $\|V_s^\sharp\|_{\rho_s} \leq e^{-s} \|V_0^\sharp\|_{\rho_0}$ and $H^{(s)}$ satisfies (29). Therefore, there exists a symplectic diffeomorphism that conjugates $H^{(s)}$ to $H^{(s+1)} = H_0 + U_{s+1} + V_{s+1}$ with

$$U_{s+1} = U_s + \Pi_{\bar{v}} V_s^{\leq N} = \sum_{j=0}^s \Pi_{\bar{v}} V_j^{\leq N}$$

and $\|V_{s+1}^\sharp\|_{\rho_{s+1}} \leq \beta_s \|V_s^\sharp\|_{\rho_s}$, with

$$\beta_s = \max\left(e^{-NxR_\varphi}, \frac{8N}{x} (\|U_s^\sharp\|_{\rho_s} + 2\|V_s^\sharp\|_{\rho_s})\right),$$

see (30). Since $NxR_\varphi \geq 1$, $\frac{8N}{x} \leq \frac{2^7}{R_\varphi} \frac{\epsilon}{\epsilon_*}$ and $\|U_s^\sharp\|_{\rho_s} + 2\|V_s^\sharp\|_{\rho_s} \leq \sum_{j=0}^s e^{-j} + e^{-s} \leq 2$, we conclude that $\beta_s \leq e^{-1}$ and H_{s+1} satisfies the induction hypothesis as well.

This proves that $(H_0 + \epsilon P) \circ \Psi = H_0 + U_r + V_r$. Since $r \geq N/c$, the remainder V_r can be written as $e^{-N/c} \epsilon F$ with F bounded as in the statement. Estimates on the resulting Hamiltonian H_r , rather than on its Hamiltonian vector field, are obtained on the basis of (31).

Finally, iterating (32) gives $\Psi - \text{id} = \sum_{s=0}^{r-1} \mathcal{O}(N\|V_s^\sharp\|_{\rho_s}) = \sum_{s=0}^{r-1} \mathcal{O}(Ne^{-s} \|V_0^\sharp\|_{\rho_s}) = \mathcal{O}(N\|V_0^\sharp\|_{\rho_0}) = \mathcal{O}(N\epsilon \|P\|_R) = \mathcal{O}(\sqrt{\epsilon/\epsilon_*})$. ■

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