



**HAL**  
open science

## Coherent states for the quantum complete rigid rotor

Daniele Fontanari, Dmitrií A Sadovskií

► **To cite this version:**

Daniele Fontanari, Dmitrií A Sadovskií. Coherent states for the quantum complete rigid rotor. *Journal of Geometry and Physics*, 2018, 129, pp.70-89. 10.1016/j.geomphys.2018.02.021 . hal-04218562

**HAL Id: hal-04218562**

**<https://ulco.hal.science/hal-04218562>**

Submitted on 22 Nov 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Coherent states for the quantum complete rigid rotor

Daniele Fontanari and Dmitrií A. Sadovskií

*Département de physique, Université du Littoral – Côte d’Opale,  
59140 Dunkerque, France*

February 21, 2018

## Abstract

Motivated by the possibility to describe orientations of quantum triaxial rigid rotors, such as molecules, with respect to both internal (body-fixed) and external (laboratory) frames, we go through the theory of coherent states and design the appropriate family of coherent states on  $T^*\text{SO}(3)$ , the classical phase space of the freely rotating rigid body (the Euler top). We pay particular attention to the resolution of identity property in order to establish the explicit relation between the parameters of the coherent states and classical phase-space variables, actions and angles.

## 1 Introduction

The concept of coherent state has been introduced in 1963 in quantum optics to describe the electromagnetic field [1, 2, 3], and since then it plays an important role in mathematical physics [4, 5, 6]. It is possible to interpret coherent states of a quantum system as the ones that are the most localized, as much as it is permitted by the Heisenberg uncertainty principle, in the corresponding classical phase space. This conceptual importance has driven several attempts to extend the definition to more general contexts [6, 7], i.e. different phase spaces. This would, in principle, allow a description of the system, and of its evolution, in terms of quantities that have classical interpretation, henceforth providing a toolbox of methods of analysis borrowed from the classical framework.

Despite many efforts, there is not, at the moment, a standard definition for a general case. Attempts to construct coherent states for a given system usually fall into few categories. It may be possible, for instance, to represent the system as a family of non-interacting harmonic oscillators, for which the definition of coherent states is straightforward, going back to the original definition in [1, 2]. In the context of geometric quantization (in the presence of a Kähler polarization), coherent states over a symplectic manifold can be also naturally defined [8, 9, 10]. A third way, see for instance [11, 12, 7], is to apply a unitary group action (on the Hilbert space of the quantum system) to a reference coherent state, and to define, following the orbits of this action, a family of coherent states. The latter are labeled as the coset space which represents naturally the classical phase space.

## 1.1 Purpose and motivation of the paper

An universal property characterizing any family of coherent states is the presence of an associated measure, see, for instance, [6]. In all the cases cited above, this measure is in direct relation to the symplectic volume of the corresponding classical system, as shown by [10] in the context of geometric quantization. The aim of the present paper is to construct, using this relation as a guideline, a family of coherent states for the *complete* non-linear rigid body (also known as the Euler top). Such body has three non-zero principal moments of inertia and is considered both in the laboratory and in the body fixed frame. Its center of mass is fixed at the origin and the body rotates freely about it. Orientations of the body are given by the elements of  $SO(3) \sim \mathbb{R}P^3$  and the phase space  $T^*SO(3)$  has dimension six. This choice of system is motivated by its importance and prevalence in molecular and nuclear physics. In particular, coherent states for this system allow quantum-classical description of the orientations of quantum rotors in the laboratory fixed frame, a topic that draws increasing attention [13, 14] in view of continuing major development of experimental techniques, see, for example, [15, 16, 17].

We like to point out that coherent states for the *reduced* rigid body with the phase space  $\mathbb{S}^2$ , the so-called spin or Bloch coherent states [18, 11], are widely known and used. Such states can describe the semi-classical evolution of the orientation of the angular momentum in the body fixed (internal) frame of the quantum rotor. The linear rigid body also has the phase space  $T^*\mathbb{S}^2$  and it has been recently studied in [19], see sec. 2.3. On the other hand, to our knowledge, only a few attempts [20, 21, 22] of constructing coherent states for the complete system exist, and none of the families of the coherent states proposed for this system relied on the associated measure in order to recover fully the parametrization of  $T^*SO(3)$ . The procedure required to build such states is far from being trivial. It employs extensively the techniques from the theory of coherent states in other contexts as well as methods developed specifically for our particular system. We obtain a new family of coherent states which is linked through its measure most directly to the underlying classical system. We investigate this connection further and validate it numerically on simple physical examples.

## 1.2 Outline of the construction

In order to construct a family of coherent states that parametrizes  $T^*SO(3)$ , we modify the standard Schwinger procedure [23, 24] for the construction of the algebra of observables using products of boson creation/annihilation operators. The original well known procedure applies to the case of the linear rigid body with classical phase space  $T^*\mathbb{S}^2$  and yields Bloch (or spin) coherent states [24, 18, 11]. By contrast, for  $T^*SO(3)$ , it is, to our knowledge, not possible to find a straightforward construction of the algebra. To overcome this difficulty, we resort to what can be seen as a quantum analogue of the Cayley-Klein parametrization of the configuration space  $SO(3)$  (an over-abundant set of parameters subjected to a constraint). A similar approach was suggested in [21, 20], and a clear analogy to the parameterization of Keplerian systems after Kustaanheimo-Stiefel regularization (see [25] and the references therein) should be equally acknowledged.

The main idea of the construction is to consider a direct product of two independent Schwinger representations. In this way, we obtain a family of coherent states parameterizing an extended phase space of dimension 8, where the notion of the body and the laboratory frames is already present, but where the norms of the angular mo-

momentum vector in the two frames can differ. The extra non-physical degree of freedom is represented by the difference of the two norms and the corresponding conjugated angle, and by the corresponding additional quantum number. The states are defined on an extended Hilbert space  $\mathcal{H}$ . The wanted family of coherent states is recovered by projecting orthogonally each of these states on the physical Hilbert subspace  $\mathcal{H}_0 \subset \mathcal{H}$ . The procedure removes automatically the extra degree of freedom and provides a parametrization of  $T^*\text{SO}(3)$ .

As can be expected, each coherent state obtained in this way is a superposition of products of two Bloch coherent states. The states in the product have the same angular momentum quantum number  $j$  and define the orientation of the angular momentum vector  $\mathbf{J}$  in the laboratory and in the body frame. The superposition is taken over different  $j$ . A general survey of such families of coherent states can be found in [22].

In order to justify, from a concrete standpoint, our construction, we compute the expectation values of several quantum observables and compare them to their classical analogues. The agreement is very good in the semi-classical limit  $\hbar \ll \|\mathbf{J}\|$ . To provide further illustration, we analyze the dynamics of our coherent states under the spherical top Hamiltonian. Again, the comparison to the corresponding classical dynamics is very clear and satisfactory.

We like to remark once more that the basis of our construction was given in [21, 20], from where we borrow some of our techniques, and in [22], where a broad class of families of “molecular” coherent states, including the ones in [21, 20], is discussed. However, our particular family of coherent states is new. What strongly motivates our definition, other than the clean procedure under which it is obtained, is the natural connection between the resolution of the identity property of the family of coherent states, the associated measure, and the geometrical (symplectic) structure of the classical phase space. This connection is not often highlighted in the literature, and we make it explicit in the present work. Our approach allows associating each our coherent state to a unique point in the phase space near which the state is localized. This correspondence is (independently) validated by the expectation values of physical quantum observables in the coherent states.

### 1.3 Structure of the paper

We begin with a brief review in sec. 2 of the three major families of coherent states, namely the Euclidean or oscillator coherent states (sec. 2.1), the Bloch or spin coherent states (sec. 2.2) and the Schwinger coherent states (sec. 2.3). Our purpose in sec. 2 is to recall concisely the techniques that we will use later in the construction of our family of coherent states. In each of these three cases, we show how the resolution of identity property can be linked to the classical symplectic volume of the underlying phase space. We demonstrate how the particular coherent states can be labeled using the coordinates on the classical phase space with which they are associated. We consider also how this labeling agrees with the expectation values of the observables of the system computed for these coherent states.

In sec. 3, we detail the construction of coherent states on  $T^*\text{SO}(3)$ . Following the outline in sec. 2, we explain in sec. 3.1 how our coherent states are labeled by the action-angle coordinates on  $T^*\text{SO}(3)$ . Subsequently, we demonstrate how such parameterization agrees with the expectation values of the observables of the system computed for such coherent states. We conclude in sec. 3.2 by relabeling our states with the coordinates which are obtained formally by the natural left or right trivialization of  $T^*\text{SO}(3)$ , and which physicists use commonly instead of the action-angle coordinates.

In sec. 4, we study the dynamics of the most simple quantum rotor system, the rigid spherical top. In the classical analog of this system, all evolution takes place in the laboratory frame. The advantage of our coherent states, by contrast to the usual Bloch states, is that they can exhibit this dynamics. We begin in sec. 4.1 by introducing Bargmann and Husimi representations of our coherent states and then turn to the concrete system. Following the evolution of the family of our states, we visualize their close relation to the classical dynamics.

## 2 Coherent states

In this section, we consider several existing definitions of coherent states, notably Euclidean (or oscillator), Bloch (or spin), and Schwinger states which we denote by  $|\mathbf{p}\rangle_{oc}$ ,  $|\mathbf{p}\rangle_{bc}$ , and  $|\mathbf{p}\rangle_{sc}$ , respectively. We review the essential concepts required for the development of the theory of the coherent states on  $T^*\text{SO}(3)$  in sec. 3. This is not intended to be a complete reference on coherent states; for a more in-depth analysis see, for instance, [4, 6]. We focus in particular on the relationship, given by the resolution of the identity property, between the family of coherent states  $|\mathbf{p}\rangle_*$  and the symplectic volume of the underlying classical phase space.

### 2.1 Euclidean coherent states

The original setting for the definition of coherent states by Klauder, Glauber and Sudarshan [3, 1, 2] was Euclidean. The underlying phase space is  $T^*\mathbb{R}$  with canonical coordinates  $(q, p)$  representing the position and momentum of a 1-dimensional particle. This space can be identified with the Euclidean 2-plane  $\mathbb{R}_{q,p}^2$  or with its complexification  $\mathbb{C}$ .

Let  $\hat{q}$  and  $\hat{p}$  denote the position and momentum operators satisfying the usual Heisenberg algebra commutation relations

$$[\hat{q}, \hat{p}] = i\hbar\mathbf{1} , \quad (1)$$

where  $\mathbf{1}$  denotes the identity operator. Define the (non self-adjoint) annihilation operator

$$\hat{a} = \frac{\hat{q} + i\hat{p}}{\sqrt{2\hbar}} \quad (2a)$$

along with its adjoint, the creation operator

$$\hat{a}^\dagger = \frac{\hat{q} - i\hat{p}}{\sqrt{2\hbar}} \quad (2b)$$

for which relation (1) becomes

$$[\hat{a}, \hat{a}^\dagger] = \mathbf{1}. \quad (3)$$

The entire Hilbert space is spanned by the eigenstates of the particle number operator  $\hat{n} = \hat{a}^\dagger\hat{a}$ . The spectrum of  $\hat{n}$  is non-degenerate and equals  $\mathbb{N}_0$  (the set of non-negative integers). For any eigenvalue  $n \in \mathbb{N}_0$ , we will denote by  $|n\rangle$  the respective eigenstate of  $\hat{n}$ , that is the state satisfying the relation

$$\hat{n}|n\rangle = n|n\rangle.$$

It follows from (3) that

$$\begin{aligned}\hat{a}^\dagger |n\rangle &= \sqrt{n+1} |n+1\rangle \\ \hat{a} |n\rangle &= \sqrt{n} |n-1\rangle \text{ for } n > 0, \quad \text{and } \hat{a}|0\rangle = 0.\end{aligned}$$

For any  $z \in \mathbb{C}$  we define the *oscillator coherent state* associated to  $z$  as the eigenstate of  $\hat{a}$  relative to the complex eigenvalue  $z$ , that is

$$\hat{a}|z\rangle_{\text{oc}} = z|z\rangle_{\text{oc}}. \quad (4)$$

Explicitly, we obtain the equivalent definition

$$|z\rangle_{\text{oc}} = e^{z\hat{a}^\dagger - \bar{z}\hat{a}} |0\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{n!} (\hat{a}^\dagger)^n |0\rangle.$$

A fundamental characteristics of this family of coherent states is given by the resolution of the identity property

$$\int_{\mathbb{C}} |z\rangle_{\text{oc}} \text{oc}\langle z| \frac{1}{\pi} d^2z = \mathbf{1}, \quad (5)$$

where  $d^2z = d\text{Re } z \times d\text{Im } z$  is the standard area element (measure) on  $\mathbb{C}$  in the identification with  $\mathbb{R}^2$  (in literature the notation  $\frac{1}{2i}dzd\bar{z}$  or  $\frac{1}{2i}dz \wedge d\bar{z}$  is also sometimes employed, it is important not to confuse the integral (5) with a contour integral in the complex plane).

Using (4) we can verify that

$$\text{oc}\langle z|\hat{a}|z\rangle_{\text{oc}} = z,$$

and that, consequently,

$$\text{oc}\langle z|\hat{a}^\dagger|z\rangle_{\text{oc}} = \overline{\text{oc}\langle z|\hat{a}|z\rangle_{\text{oc}}} = \bar{z}.$$

Expressing  $\hat{p}$  and  $\hat{q}$  in terms of operators (2), it immediately follows that

$$\begin{aligned}\text{oc}\langle z|\hat{q}|z\rangle_{\text{oc}} &= \sqrt{2\hbar} \frac{z + \bar{z}}{2} = \sqrt{2\hbar} \text{Re } z \\ \text{oc}\langle z|\hat{p}|z\rangle_{\text{oc}} &= \sqrt{2\hbar} \frac{z - \bar{z}}{2i} = \sqrt{2\hbar} \text{Im } z.\end{aligned}$$

This shows that it is natural to identify the complex space  $\mathbb{C}$  of parameters  $z$  with the real plane  $\mathbb{R}_{q,p}^2$  with coordinates

$$q = \sqrt{2\hbar} \text{Re } z \quad \text{and} \quad p = \sqrt{2\hbar} \text{Im } z,$$

such that

$$z = \frac{q + ip}{\sqrt{2\hbar}}. \quad (6)$$

Equation (6) gives real parameterization of the coherent states  $|z\rangle_{\text{oc}} = |q, p\rangle_{\text{oc}}$  for which identity (5) becomes

$$\int_{\mathbb{R}^2} |q, p\rangle_{\text{oc}} \text{oc}\langle q, p| \frac{dq dp}{2\pi\hbar} = \mathbf{1}.$$

### 2.1.1 Multi-dimensional oscillator coherent states

The above construction can be trivially extended (via tensor product) to the case with several degrees of freedom. Let  $\hat{q}_i$  and  $\hat{p}_i$  with  $i = 1, \dots, M$  be now the position/momentum components operators and consider the respective canonical variables  $q, p \in \mathbb{R}^M$  (or, equivalently,  $z = \frac{q+ip}{\sqrt{2\hbar}} \in \mathbb{C}^M$ ) of an  $M$  degrees of freedom Hamiltonian dynamical system. A coherent state  $|q, p\rangle_{\text{oc}}$  (defined, as before, as an eigenstate of the  $M$  commuting annihilation operators) satisfies now

$${}_{\text{oc}}\langle q, p | \hat{q}_i | q, p \rangle_{\text{oc}} = q_i \quad \text{and} \quad {}_{\text{oc}}\langle q, p | \hat{p}_i | q, p \rangle_{\text{oc}} = p_i,$$

while the resolution of the identity property becomes

$$\int_{\mathbb{R}^{2M}} |q, p\rangle_{\text{oc}} {}_{\text{oc}}\langle q, p | \frac{d^M q d^M p}{(2\pi\hbar)^M} = \mathbf{1}. \quad (7)$$

Note that it is possible to express (7) in a nice coordinate-free way. The classical phase space  $T^*\mathbb{R}^M \simeq \mathbb{R}_{q,p}^{2M}$  (parameterized here by  $q$  and  $p$ ) is equipped with a canonical symplectic 2-form  $\omega = dq \wedge dp$ . Denoting the  $M$ -th exterior power of  $\omega$ , which is identified in a natural way with a *density* on  $\mathbb{R}^{2M}$ , as  $\omega^M$  we come to

$$\int_{\mathbb{R}^{2M}} |q, p\rangle_{\text{oc}} {}_{\text{oc}}\langle q, p | \left( \frac{\omega(q, p)}{2\pi\hbar} \right)^M = \mathbf{1}.$$

## 2.2 Bloch coherent states

Bloch (or spin) coherent states [18, 26], are generally employed when the classical phase space is a 2 dimensional sphere  $\mathbb{S}^2$  equipped with the Poisson algebra  $\mathfrak{su}(2)$  of the angular momentum. In particular, we can think of a point particle which has non-zero spin and is placed at the origin in  $\mathbb{R}^3$ . Spin is an internal angular momentum  $\mathbf{J} = (J_1, J_2, J_3)$  of fixed length  $|\mathbf{J}|$  whose orientations in  $\mathbb{R}^3$  are given by points  $\mathbf{p}$  of  $\mathbb{S}^2$  (the Bloch sphere). In quantum optics, points of  $\mathbb{S}^2$  define polarizations of the photon. In our context, the natural example is provided by the *reduced Euler top* system, where we study orientations of the angular momentum  $\mathbf{J}$  of the freely rotating rigid body only with regards to the body-fixed frame.

The basic idea of the construction of Bloch coherent states dates back to Gilmore [11] and Perelomov [12], see also [6, 7]. This family can also be naturally recovered in the framework of geometric quantization, see [10]. The symmetry group of the system is  $SU(2)$ . We choose an unitary irreducible representation, of highest (integer or half-integer) weight  $j$ , of  $SU(2)$  hosted on the complex  $2j + 1$  dimensional Hilbert space  $\mathcal{H}_j \ni |\mathbf{p}\rangle$  and consider the natural action of  $SU(2)$  on the space  $\mathbb{S}^2 \ni \mathbf{p}$ . Specifically, the action of any element  $R \in SU(2)$  is represented by an unitary matrix  $U_R$  acting on  $\mathcal{H}_j$ . Let  $G_{\mathbf{p}} \subset SU(2)$  be the stabilizer of the point  $\mathbf{p}$ , i.e.,  $\mathbf{p}$  remains invariant under the action of  $G_{\mathbf{p}}$ . The coherent state  $|\mathbf{p}\rangle$ , corresponding to  $\mathbf{p}$ , is chosen to be invariant, up to a phase factor, under the action (in the representation) of any element in  $G_{\mathbf{p}}$ . Furthermore,  $|R\mathbf{p}\rangle = U_R|\mathbf{p}\rangle$  defines the coherent state localized around the point  $R\mathbf{p}$  for any  $R \in SU(2)$ . Since the action of  $SU(2)$  is transitive, the family of coherent states defined in this way covers the whole sphere and if  $R_1\mathbf{p} = R_2\mathbf{p}$  then  $|R_1\mathbf{p}\rangle$  and  $|R_2\mathbf{p}\rangle$  differ only by a phase factor. In this way the problem of defining the family of coherent states is well posed and depends only on the choice of  $|\mathbf{p}\rangle$  for a given reference point  $\mathbf{p}$ . It is straightforward to see that  $|\mathbf{p}\rangle$  must be an eigenstate of  $\mathbf{p} \cdot \hat{\mathbf{J}}$

( $\hat{\mathbf{J}}$  being the vector whose components are the angular canonical angular momentum projections operators, see below for further details). In order to fix the choice (among the  $2j + 1$  possible candidates) it is customary to require for  $|\mathbf{p}\rangle$  to minimize a specific uncertainty relation: if  $\mathbf{q} \in \mathbb{S}^2$  is orthogonal to  $\mathbf{p}$ , then  $|\mathbf{p}\rangle$  should be chosen in a way that minimize the variance of  $\mathbf{q} \cdot \hat{\mathbf{J}}$ . A straightforward calculation shows that the choice of  $|\mathbf{p}\rangle$  as the eigenstate  $\mathbf{p} \cdot \hat{\mathbf{J}}$  relative to the highest eigenvalue, guarantees this condition (independently from  $\mathbf{q}$ , and always with the same variance  $\hbar^2 j/2$ ). This can be interpreted as  $|\mathbf{p}\rangle$  being, classically, localized around the point  $\mathbf{p}$ .

Let  $\hat{J}_i$ , with  $i = 1, 2, 3$ , denote the canonical generators (represented by self-adjoint operators acting on  $\mathcal{H}_j$ ) of the Lie algebra  $\mathfrak{su}(2)$  satisfying

$$[\hat{J}_i, \hat{J}_j] = i\hbar\epsilon_{ijk}\hat{J}_k \quad i, j = 1, 2, 3,$$

and let  $|j, m\rangle$ ,  $m = -j, \dots, j$ , be the elements of the canonical base of  $\mathcal{H}_j$ , such that

$$\hat{J}_3 |j, m\rangle = \hbar m |j, m\rangle.$$

The Bloch coherent state (of spin  $j$ ) relative to the parameter  $z \in \mathbb{C}$  is defined as

$$|j, z\rangle_{\text{bc}} = \frac{1}{(1 + |z|^2)^j} \sum_{m=-j}^j \sqrt{\binom{2j}{j+m}} \bar{z}^{j+m} |j, m\rangle. \quad (8)$$

While the elements of this family are not eigenstates of any annihilation operator (this would be *a priori* impossible due to the finite-dimensionality of  $\mathcal{H}_j$ ), they satisfy the resolution of identity property

$$\int_{\mathbb{C}} |j, z\rangle_{\text{bc}} \langle j, z| \frac{1 + 2j}{\pi(1 + |z|^2)^2} d^2z = \mathbf{1}. \quad (9)$$

To interpret the situation from the classical point of view, consider the sphere  $\mathbb{S}_\rho^2$  of radius  $\rho = \hbar(j + 1/2)$  embedded in  $\mathbb{R}^3$  through the identification

$$\mathbb{S}_\rho^2 = \{\mathbf{p} = (J_1, J_2, J_3) \in \mathbb{R}^3 \mid \|\mathbf{p}\|^2 = J_1^2 + J_2^2 + J_3^2 = \rho^2\}.$$

The space  $\mathbb{S}_\rho^2$  is equipped with a natural symplectic 2-form  $\omega$  which equals the canonical (oriented) area form rescaled by a factor of  $\rho^{-1}$  (in this way  $(\arg(J_1 + iJ_2), J_3)$ , the cylindrical coordinates with respect to the third axis on the sphere, are canonical). Introducing complex stereographical coordinates on  $\mathbb{S}_\rho^2 \setminus (0, 0, \rho)$  (excluding the North pole), also known as Riemann projection,

$$z = \frac{J_1 + iJ_2}{\rho - J_3}, \quad (10)$$

we can rewrite the resolution of identity in the familiar form

$$\int_{\mathbb{C}} |j, z\rangle_{\text{bc}} \langle j, z| \frac{\omega(z)}{2\pi\hbar} = \mathbf{1}.$$

Moreover, if  $\mathbf{p}$  and  $z$  are related by (10) we have that

$$\langle j, z | \hat{J}_i | j, z \rangle_{\text{bc}} = \frac{\hbar j}{\rho} J_i = J_i + O(\hbar).$$

These two formulas allow us to conclude that  $|j, z\rangle_{\text{bc}}$  can be associated, in the semi-classical limit, to the point  $\mathbf{p}$  on the sphere of radius  $\rho$  where  $\mathbf{p}$  and  $z$  are related through (10). We can also observe that the choice of  $\rho$  agrees up to first order in  $\hbar$  to the quantum value of  $\|\hat{\mathbf{J}}\| = \hbar\sqrt{j(j+1)}$  and it is the value which is often encountered in semi-classical analysis.

### 2.3 Schwinger coherent states

Schwinger coherent states [24] generalize in some sense the construction of Bloch coherent states as seen in sec. 2.2 while taking advantage of the creation/annihilation scheme seen in sec. 2.1. This is made possible thanks to the well known explicit correspondence between the 1:1 resonant oscillator system with dynamical symmetry  $SU(2)$  and the reduced Euler top system described by the Bloch coherent states [23, 27], see also [28, 29]. Specifically, as detailed by Schwinger in [23], we can rewrite the elements generating the  $so(3)$  algebra of quantum observables  $\hat{J}_i$ , the components of the angular momentum, as quadratic functions of two commuting pairs of creation/annihilation operators representing the two modes of the quantum 1:1 resonant oscillator (note that this boson representation generalizes to any finite dimensional Lie algebra). Subsequently, oscillator coherent states are introduced as in the Euclidean case in sec. 2.1. When classified in terms of angular momentum (or spin) quantum number  $j$ , these states turn out to be a weighted superposition of Bloch states  $|j, \mathbf{p}\rangle_{bc}$  with different  $j$  but with the same orientation of  $\mathbf{p} = (J_1, J_2, J_3)$ . All values of  $j$  are present and the weights are themselves controlled by a parameter which allows to peak on the desired magnitude of the angular momentum (instead of having states of fixed  $j$  like the Bloch coherent states). The corresponding classical two-degree-of-freedom system can be described as rotation of a linear top, i.e., of a rigid body with one zero principal moment of inertia. Our construction may differ from the ones implemented elsewhere, notably in nuclear physics. Our interest here is to prepare for a similar projective procedure in the higher-dimensional case of  $T^*SO(3)$  in sec. 3. For a complete discussion of other constructions the reader may refer to [6], see also [19].

#### 2.3.1 Two-oscillator (boson) coherent states

We detail the construction. Continuing sec. 2.1.1, consider a two-mode ( $M = 2$ ) oscillator system and let  $\hat{a}_1$  and  $\hat{a}_2$  be its two commuting annihilation operators,

$$[\hat{a}_1, \hat{a}_2] = [\hat{a}_1, \hat{a}_2^\dagger] = \mathbf{0}, \quad [\hat{a}_i, \hat{a}_i^\dagger] = \mathbf{1}, \quad i = 1, 2.$$

The number of quanta in each mode is given by

$$\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i, \quad i = 1, 2.$$

The Hilbert space is constructed naturally as a direct product  $\mathcal{H}_1 \times \mathcal{H}_2$  with basis functions  $|n_1, n_2\rangle := |n_1\rangle |n_2\rangle$  such that

$$\hat{n}_i |n_1, n_2\rangle = n_i |n_1, n_2\rangle, \quad i = 1, 2$$

where  $n_1, n_2 \in \mathbb{N}_0$ , and

$$\begin{aligned} \hat{a}_1 |n_1, n_2\rangle &= \sqrt{n_1} |n_1 - 1, n_2\rangle \\ \hat{a}_1^\dagger |n_1, n_2\rangle &= \sqrt{n_1 + 1} |n_1 + 1, n_2\rangle \\ \hat{a}_2 |n_1, n_2\rangle &= \sqrt{n_2} |n_1, n_2 - 1\rangle \\ \hat{a}_2^\dagger |n_1, n_2\rangle &= \sqrt{n_2 + 1} |n_1, n_2 + 1\rangle. \end{aligned}$$

We can now introduce standard (oscillator) coherent states  $|z_1, z_2\rangle_{oc}$  with  $z_{1,2} \in \mathbb{C}$  as

$$\hat{a}_i |z_1, z_2\rangle_{oc} = z_i |z_1, z_2\rangle_{oc}, \quad i = 1, 2$$

and obtain their explicit form (cf sec. 2.1)

$$|z_1, z_2\rangle_{\text{oc}} = e^{-\frac{1}{2}(|z_1|^2 + |z_2|^2)} \sum_{n_1, n_2=0}^{\infty} \frac{z_1^{n_1} z_2^{n_2}}{\sqrt{n_1! n_2!}} |n_1, n_2\rangle. \quad (11)$$

By construction, functions  $|z_1, z_2\rangle_{\text{oc}}$  are normalized, and resolve identity

$$\int_{\mathbb{C}^2} |z_1, z_2\rangle_{\text{oc}} \langle z_1, z_2| \frac{d^2 z_1 d^2 z_2}{\pi^2} = \mathbf{1}.$$

### 2.3.2 Schwinger coherent states

The SU(2) invariance of the system implies that the energy commutes with the (rescaled) total number of quanta operator

$$\hat{I} = \frac{\hbar}{2}(\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2) = \frac{\hbar}{2}(\hat{n}_1 + \hat{n}_2). \quad (12)$$

(In classical mechanics, the corresponding momentum is the first integral which is in involution with the Hamiltonian and which induces an  $\mathbb{S}^1$  action on the classical phase space  $\mathbb{T}^*\mathbb{R}^2 \sim \mathbb{C}^2$ ). The algebra of operators that commute with  $\hat{I}$  is spanned by  $\hat{I}$  itself and three other quadratic operators

$$\begin{aligned} \hat{J}_1 &= \frac{\hbar}{2}(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1), \\ \hat{J}_2 &= \frac{\hbar}{2i}(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1), \\ \hat{J}_3 &= \frac{\hbar}{2}(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2) = \frac{\hbar}{2}(\hat{n}_1 - \hat{n}_2), \end{aligned}$$

which generate the  $\mathfrak{su}(2) \sim \mathfrak{so}(3)$  Lie algebra

$$[\hat{J}_i, \hat{J}_j] = i\hbar\epsilon_{ijk}\hat{J}_k, \quad [\hat{J}_i, \hat{I}] = \mathbf{0}, \quad \text{for } i, j = 1, 2, 3,$$

with Casimir  $\hat{I}$ . We can show, by direct calculation, that

$$\hat{J}^2 := \hat{\mathbf{J}} \cdot \hat{\mathbf{J}} = \hat{J}_1^2 + \hat{J}_2^2 + \hat{J}_3^2 = \hat{I}(\hat{I} + \hbar\mathbf{1}).$$

We can consider joint eigenstates of  $\hat{J}_3$  and  $\hat{I}$  (or equivalently  $\hat{J}^2$ ) as an alternative basis for the Hilbert space. Specifically, let us define the states  $|j, m\rangle_s$  with  $2j \in \mathbb{N}_0$ ,  $j + m \in \mathbb{Z}$ , and  $|m| \leq j$ , such that

$$\begin{aligned} \hat{I}|j, m\rangle_s &= \hbar j|j, m\rangle_s \quad \text{or} \quad \hat{J}^2|j, m\rangle_s = \hbar^2 j(j+1)|j, m\rangle_s, \quad \text{and} \\ \hat{J}_3|j, m\rangle_s &= \hbar m|j, m\rangle_s. \end{aligned}$$

It follows from definitions of  $\hat{I}$  and  $\hat{J}_3$  that  $|j, m\rangle_s = |n_1, n_2\rangle$  provided that

$$n_1 = j + m \quad \text{and} \quad n_2 = j - m.$$

We can rewrite oscillator coherent state  $|z_1, z_2\rangle_{\text{oc}}$  in (11) in the basis given by  $|j, m\rangle_s$ . This can be most conveniently accomplished after introducing two complex parameters

$$\alpha = (1 + |z_1 z_2^{-1}|^2)^{-\frac{1}{2}} \bar{z}_2^2 \quad \text{and} \quad \beta = \bar{z}_1 \bar{z}_2^{-1},$$

whose physical meaning will become clear in sec. 2.3.4. Using new parameters  $(\alpha, \beta)$  and relabeling  $|z_1, z_2\rangle_{\text{oc}}$  as  $|\alpha, \beta\rangle_{\text{sc}}$ , we introduce Schwinger coherent states

$$|\alpha, \beta\rangle_{\text{sc}} = e^{-\frac{1}{2}|\alpha|} \sum_{2j \in \mathbb{N}_0} \frac{\bar{\alpha}^j}{\sqrt{(2j)!}} (1 + |\beta|^2)^{-j} \sum_{m=-j}^j \sqrt{\binom{2j}{j+m}} \bar{\beta}^{j+m} |j, m\rangle_{\text{s}}.$$

In terms of Bloch coherent states (8) in sec. 2.2, the above expression for  $|z_1, z_2\rangle_{\text{oc}} = |\alpha, \beta\rangle_{\text{sc}}$  takes a particularly simple compact form

$$|\alpha, \beta\rangle_{\text{sc}} = e^{-\frac{1}{2}|\alpha|} \sum_{2j \in \mathbb{N}_0} \frac{\bar{\alpha}^j}{\sqrt{(2j)!}} |j, \beta\rangle_{\text{bc}}. \quad (13)$$

In the  $(\alpha, \beta)$  parametrization, the resolution of the identity becomes

$$\int_{\mathbb{C}^2} |\alpha, \beta\rangle_{\text{sc}} \langle \alpha, \beta| \frac{d^2\alpha d^2\beta}{2\pi^2(1 + |\beta|^2)^2} = \mathbf{1}. \quad (14)$$

### 2.3.3 Coherent states on subspaces of integer $j$

We modify our construction further. The Hilbert space  $\mathcal{H}$  of Schwinger coherent states (13) is a direct sum  $\mathcal{H} = \mathcal{H}' \oplus \mathcal{H}''$  of subspaces spanned by functions  $|j, m\rangle_{\text{s}}$  with integer and half-integer  $j$ , respectively. Often the interest lies in the subspace  $\mathcal{H}' \subset \mathcal{H}$  alone hosting true (non-projective) representations of  $\text{SO}(3)$  rather than those of its double cover  $\text{SU}(2)$ .

While it is not possible to construct coherent states on  $\mathcal{H}'$  in the standard way, we can consider such states as eigenstates of “modified” commuting annihilation operators

$$\hat{A}_1 = \hat{a}_1 \hat{a}_1 \quad \text{and} \quad \hat{A}_2 = \hat{a}_1 \hat{a}_2$$

which are well defined on  $\mathcal{H}'$ . Specifically, their action on  $|j, m\rangle_{\text{s}} \in \mathcal{H}'$  is

$$\begin{aligned} \hat{A}_1 |j, m\rangle_{\text{s}} &= \sqrt{(j+m)(j+m-1)} |j-1, m-1\rangle_{\text{s}} \\ \hat{A}_2 |j, m\rangle_{\text{s}} &= \sqrt{(j+m)(j-m)} |j-1, m\rangle_{\text{s}}. \end{aligned}$$

Each joint eigenspace of these operators is doubly degenerate and contains exactly one state  $|\alpha, \beta\rangle_{\text{sc}_2}$  which belongs to  $\mathcal{H}'$ . It is immediate to see that  $|\alpha, \beta\rangle_{\text{sc}_2}$  is the orthogonal projection of  $|\alpha, \beta\rangle_{\text{sc}}$  in (13) on  $\mathcal{H}'$ . Taking normalization into account and using the same notation as before in (13), we obtain

$$|\alpha, \beta\rangle_{\text{sc}_2} = (\cosh |\alpha|)^{-\frac{1}{2}} \sum_{j \in \mathbb{N}_0} \frac{\bar{\alpha}^j}{\sqrt{(2j)!}} |j, \beta\rangle_{\text{bc}} \in \mathcal{H}' \quad (15)$$

which is the equivalent of equation (14) in [19].

Observing that on the integer  $j$  subspaces

$$|\alpha, \beta\rangle_{\text{sc}} \langle \alpha, \beta|_{\mathcal{H}'} = \frac{1 + e^{-2|\alpha|}}{2} |\alpha, \beta\rangle_{\text{sc}_2} \langle \alpha, \beta|_{\text{sc}_2},$$

and that the measure defining the resolution of the identity should be rescaled accordingly, the resolution of identity, understood now as acting only on  $\mathcal{H}' \subset \mathcal{H}$ , is

$$\int_{\mathbb{C}^2} |\alpha, \beta\rangle_{\text{sc}_2} \langle \alpha, \beta| \frac{1 + e^{-2|\alpha|}}{4\pi^2(1 + |\beta|^2)^2} d^2\alpha d^2\beta = \mathbf{1}. \quad (16)$$

### 2.3.4 The classical choice of coherent state parameters

We uncover the physical meaning of our parameters  $\alpha$  and  $\beta$  and, subsequently, we rely on the resolution of identity (16) to establish the relationship between the family of coherent states  $|\alpha, \beta\rangle_{\text{sc}_2}$  defined in (15) and the geometry of the classical phase space. The expression of  $|\alpha, \beta\rangle_{\text{sc}_2}$  in terms of Bloch coherent states suggests that the classical degree of freedom, corresponding to the parameter  $\beta$ , should be represented by a sphere. As seen in sec. 2.2, the radius of the Bloch sphere is related to the value of quantum number  $j$ . In the case of  $|\alpha, \beta\rangle_{\text{sc}_2}$  we have a family of Bloch states centered on what we call the target norm of classical angular momentum  $J$ . Since the weights of the Bloch coherent states in (15) are controlled by  $\alpha$  it is reasonable to impose the ansatz  $\alpha = a(J) e^{i\phi_J}$  where  $a(J) > 0$  is an increasing function of  $J$  representing the classical radius of the sphere parametrized by  $\beta$ . Requiring furthermore that  $J$  and  $\phi_J$  are canonically conjugated variables with respect to the classical symplectic structure, and comparing to (9), we obtain that the measure defining the resolution of identity (16) becomes

$$\frac{1 + e^{-2|\alpha|}}{4\pi^2(1 + |\beta|^2)^2} d^2\alpha d^2\beta = \hbar^2 \frac{1 + e^{-2a(J)}}{4J} a(J) \frac{da(J)}{dJ} \left( \frac{\omega}{2\pi\hbar} \right)^2, \quad (17)$$

where we used the fact that  $\omega^2 = \omega_1 \omega_2$ , being (with some abuse of notation, identifying 2-form on 2-dimensional manifolds as densities)  $\omega_1 = dJ d\phi_J$  is the symplectic volume in the canonical  $(\phi_J, J)$  coordinates, and

$$\omega_2 = \frac{4J d^2\beta}{(1 + |\beta|^2)^2}$$

the symplectic volume on the sphere of radius  $J$  parametrized, in complex stereographical coordinates, by  $\beta$ . We like to comment here that the introduction of  $a(J)$  is a rather general idea that can be applied in most cases. On the other hand, the requirement about the canonical conjugation between  $\phi_J$  and  $J$  is quite specific. It is rather natural in our opinion and it leads to a consistent theory, and to most straightforward relation between of the coherent states and the underlying classical phase space.

In analogy with the previous examples we now require the resolution of the identity measure, (17), to be equal to  $\left( \frac{\omega}{2\pi\hbar} \right)^2$ , which gives:

$$\hbar^2 \frac{1 + e^{-2a(J)}}{4J} a(J) \frac{da(J)}{dJ} = 1.$$

This is solved if we express the inverse of  $a(J)$  as

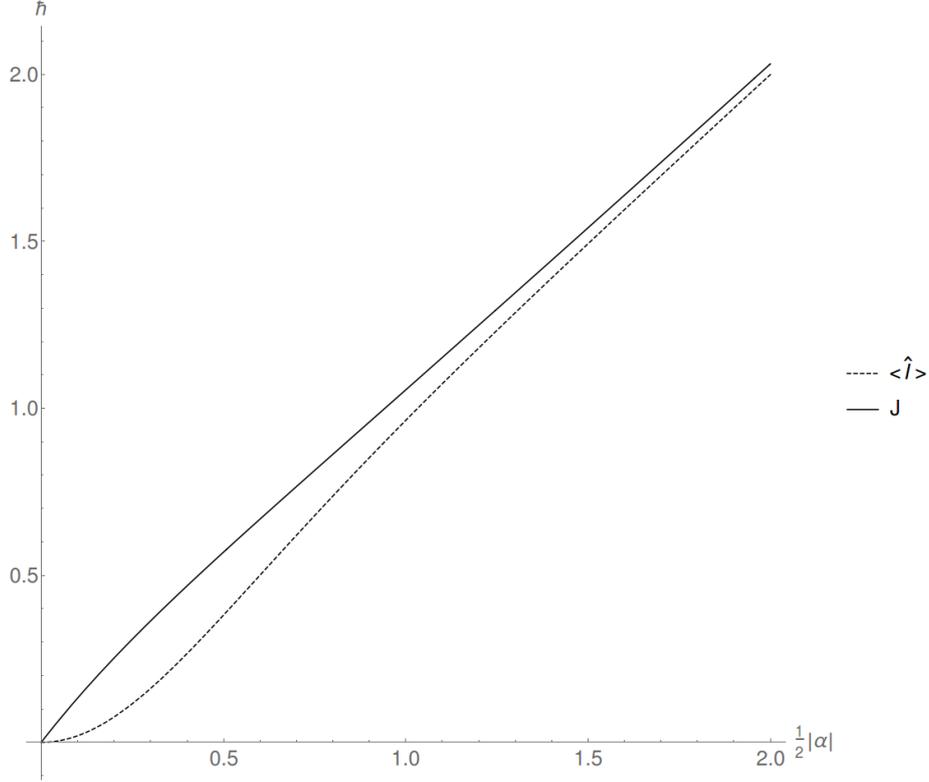
$$J(a) = \frac{\hbar}{2} \sqrt{\frac{1 + 2a^2 - (1 + 2a) e^{-2a}}{2}}.$$

We have, in this way, that  $J(a)$  is invertible, positive, increasing and such that  $J(0) = 0$ , and the same holds for  $a(J)$ .

We can better motivate now our choice of  $J$  as the radius of the Bloch sphere parametrized by  $\beta$ . We consider the expectation value of the angular momentum components on the coherent states. We have, as illustrated in figure 1,

$${}_{\text{sc}_2}\langle \alpha, \beta | \hat{I} | \alpha, \beta \rangle_{\text{sc}_2} = \frac{\hbar}{2} |\alpha| \tanh |\alpha| \approx J(|\alpha|)$$

Figure 1: Comparison between the expectation value of  $\hat{I}$  compared to  $J(|\alpha|)$  for coherent states in the Schwinger representation.



and consequently

$${}_{\text{sc}_2}\langle \alpha, \beta | \hat{J}_i | \alpha, \beta \rangle_{\text{sc}_2} \approx J_i$$

where, following again (10), we introduced the classical parameters  $(J_1, J_2, J_3) \in \mathbb{R}^3$  satisfying

$$\beta = \frac{J_1 + iJ_2}{J(|\alpha|) - J_3} \text{ and } \sqrt{J_1^2 + J_2^2 + J_3^2} = J(|\alpha|). \quad (18)$$

In order to better understand the underlying classical structure and interpret  $J_1$ ,  $J_2$  and  $J_3$ , we recall the basic aspects of the linear rigid body system (with one fixed point). This system is completely defined once the angular momentum vector  $\mathbf{J}$  and the orientation of the body are fixed. The orientation of the body can be represented by a unit vector  $\mathbf{r}$ , directed as the symmetry axis of the body (a preferential direction should be chosen). It should be observed that  $\mathbf{J} \cdot \mathbf{r} = 0$ , i.e., the angular momentum is perpendicular to the axis of symmetry of the body. Let us fix an inertial orthonormal reference frame  $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$  such that  $\mathbf{e}_z$  is not aligned with  $\mathbf{J}$ . Let  $\Sigma_z$  be the plane orthogonal to  $\mathbf{e}_z$ . We can introduce the angle  $\phi_z$  between the projection of  $\mathbf{J}$  on  $\Sigma_z$  and the  $\mathbf{e}_y$  axis. If  $J = \|\mathbf{J}\|$  denotes the length of the angular momentum vector and  $J_z = \mathbf{J} \cdot \mathbf{e}_z$  the projection of the angular momentum on axis  $\mathbf{e}_z$ , we have that the triple  $(J, J_z, \phi_z)$  determines  $\mathbf{J}$  completely through the formula

$$\mathbf{J} = -\sqrt{J^2 - J_z^2} \sin \phi_z \mathbf{e}_x + \sqrt{J^2 - J_z^2} \cos \phi_z \mathbf{e}_y + J_z \mathbf{e}_z.$$

We can finally define the angle  $\phi_J$  between  $\mathbf{r}$  and  $\Sigma_z$ . This angle is measured in the plane orthogonal to  $\mathbf{J}$  and allows to completely recover  $\mathbf{r}$  through the expression

$$\begin{aligned} \mathbf{r} = & \left( \frac{J - J_z}{2J} \cos(\phi_z - \phi_J) + \frac{J + J_z}{2J} \cos(\phi_z + \phi_J) \right) \mathbf{e}_x + \\ & + \left( \frac{J - J_z}{2J} \sin(\phi_z - \phi_J) + \frac{J + J_z}{2J} \sin(\phi_z + \phi_J) \right) \mathbf{e}_y + \\ & - \sqrt{1 - \left( \frac{J_z}{J} \right)^2} \sin(\phi_J) \mathbf{e}_z. \end{aligned}$$

In order to draw the parallel to the quantum case, we observe that  $J$  and  $\phi_J$ , as well as  $J_z$  and  $\phi_z$ , are conjugate variables, and that  $(J_z, \phi_z)$  parametrize a sphere of radius  $J$ , which can be identified with the Bloch sphere. In this way we can identify the parameters  $J_1, J_2$  and  $J_3$ , encountered in (18), with the components of the angular momentum  $\mathbf{J} \cdot \mathbf{e}_x, \mathbf{J} \cdot \mathbf{e}_y$  and  $\mathbf{J} \cdot \mathbf{e}_z$  respectively. This allows us to conclude that the Schwinger coherent states  $|\alpha, \beta\rangle_{sc_2}$  can be seen as parameterizing the phase space of the linear rigid body.

### 3 Coherent states on $T^*\text{SO}(3)$

The phase space of the non-linear rigid body  $T^*\text{SO}(3)$  is a 6-dimensional symplectic manifold. Consequently, following the general outline in sec. 2, we may expect the family of coherent states for the corresponding quantum system to be parameterized by three complex parameters. The situation is, however, more complicated because defining quantum observables as quadratic expressions of three commuting pairs of eventually modified creation/annihilation operators is not possible on  $T^*\text{SO}(3)$ .

We modify our approach. We note that components of the angular momentum  $\mathbf{J}$  in the body frame and in the laboratory frame satisfy *separately* the standard commutation relations of  $\mathfrak{so}(3)$ . So we may consider  $\mathfrak{so}(3) \times \mathfrak{so}(3)$  with one obvious constraint: the norm  $\|\mathbf{J}\|$  of the angular momentum is the same in both of the frames. This gives the idea [20, 21] to define coherent states using the Schwinger boson construction in sec. 2.3.2 for the unconstrained system on a larger Hilbert space, and then recover physical states by means of orthogonal projection (cf sec. 2.3.3). So the procedure is similar to the lower dimensional case. However now the 4-dimensional oscillator coherent state is first projected on the true Hilbert space corresponding to  $T^*\text{SO}(3)$  and then rewritten as a superposition of functions transforming according to the irreducible representations of  $\text{SO}(3)$ , specifically Wigner  $D$ -functions  $|j, k, m\rangle$  with all values of  $j, k$  and  $m$  present. The weights in this superposition are specific to  $T^*\text{SO}(3)$  and are of primary interest to concrete applications in the dynamical study of the rigid body. The construction is unambiguous and the resulting expression (22) depends only on the parameterization. We justify our choice of parameters in sec. 3.2, where we show how they relate to the orientation of the rigid body.

Consider four independent boson annihilation operators  $\hat{a}_i$  satisfying relations

$$[\hat{a}_i, \hat{a}_j] = \mathbf{0} \quad \text{and} \quad [\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij} \mathbf{1} \quad \text{with } i, j = 1, \dots, 4$$

along with the corresponding self-adjoint number operators

$$\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i \quad \text{with } i = 1, \dots, 4.$$

whose joint eigenstates span the Hilbert space  $\mathcal{H}$  of quantum states (see sec. 2.1.1). Specifically, for each  $n = (n_1, n_2, n_3, n_4) \in \mathbb{N}_0^4$  we define  $|n\rangle \in \mathcal{H}$  such that

$$\hat{n}_i |n\rangle = n_i |n\rangle \quad \text{for } i = 1, \dots, 4,$$

and denoting the  $i$ -th canonical base element of  $\mathbb{N}_0^4$  as  $1_i$  we can write that

$$\begin{aligned} \hat{a}_i |n\rangle &= \sqrt{n_i} |n - 1_i\rangle \\ \hat{a}_i^\dagger |n\rangle &= \sqrt{n_i + 1} |n + 1_i\rangle \quad \text{for } i = 1, \dots, 4. \end{aligned}$$

We can introduce quadratic operators

$$\begin{aligned} \hat{I} &= \frac{\hbar}{4} (\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + \hat{a}_3^\dagger \hat{a}_3 + \hat{a}_4^\dagger \hat{a}_4) = \frac{\hbar}{4} (\hat{n}_1 + \hat{n}_2 + \hat{n}_3 + \hat{n}_4), \\ \hat{\Lambda} &= \frac{\hbar}{4} (\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 - \hat{a}_3^\dagger \hat{a}_3 - \hat{a}_4^\dagger \hat{a}_4) = \frac{\hbar}{4} (\hat{n}_1 + \hat{n}_2 - \hat{n}_3 - \hat{n}_4), \end{aligned} \quad (19a)$$

and

$$\begin{aligned} \hat{M}_1 &= \frac{\hbar}{2} (\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1), & \hat{K}_1 &= \frac{\hbar}{2} (\hat{a}_4^\dagger \hat{a}_3 + \hat{a}_3^\dagger \hat{a}_4), \\ \hat{M}_2 &= \frac{\hbar}{2i} (\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1), & \hat{K}_2 &= \frac{\hbar}{2i} (\hat{a}_4^\dagger \hat{a}_3 - \hat{a}_3^\dagger \hat{a}_4), \\ \hat{M}_3 &= \frac{\hbar}{2} (\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2), & \hat{K}_3 &= \frac{\hbar}{2} (\hat{a}_3^\dagger \hat{a}_3 - \hat{a}_4^\dagger \hat{a}_4), \end{aligned} \quad (19b)$$

which satisfy the following commutation relations for  $i, j = 1, \dots, 4$

$$\begin{aligned} [\hat{I}, \hat{\Lambda}] &= \mathbf{0}, \\ [\hat{M}_i, \hat{K}_j] &= \mathbf{0}, \\ [\hat{M}_i, \hat{I}] &= [\hat{M}_i, \hat{\Lambda}] = \mathbf{0}, & [\hat{K}_i, \hat{I}] &= [\hat{K}_i, \hat{\Lambda}] = \mathbf{0}, \\ [\hat{M}_i, \hat{M}_j] &= i\hbar \sum_{k=1}^3 \epsilon_{ijk} \hat{M}_k, & [\hat{K}_i, \hat{K}_j] &= -i\hbar \sum_{k=1}^3 \epsilon_{ijk} \hat{K}_k \end{aligned} \quad (20)$$

and algebraic identities

$$\begin{aligned} \hat{M}^2 &= \hat{M}_1^2 + \hat{M}_2^2 + \hat{M}_3^2 = (\hat{I} + \hat{\Lambda})(\hat{I} + \hat{\Lambda} + \hbar \mathbf{1}), \\ \hat{K}^2 &= \hat{K}_1^2 + \hat{K}_2^2 + \hat{K}_3^2 = (\hat{I} - \hat{\Lambda})(\hat{I} - \hat{\Lambda} + \hbar \mathbf{1}). \end{aligned}$$

This results in constructing an  $\mathfrak{so}(3) \times \mathfrak{so}(3)$  algebra as a product of two Schwinger representations. Note that (20) are relations which we expect from the components of the angular momentum in the body frame ( $K$ ) and in the laboratory frame ( $M$ ). The equivalence to the rigid body system will be complete provided that we project on the states in  $\mathcal{H}_0 \subset \mathcal{H}$  for which  $\hat{\Lambda} = 0$ . We can re-parameterize the basis elements of  $\mathcal{H}$  using the eigenvalues of the complete set of observables  $\hat{I}, \hat{\Lambda}, \hat{M}_3, \hat{K}_3$ , and we introduce the family of states  $|j, \lambda, k, m\rangle_{s_2}$  such that

$$\begin{aligned} \hat{I} |j, \lambda, k, m\rangle_{s_2} &= \hbar j |j, \lambda, k, m\rangle_{s_2}, \\ \hat{\Lambda} |j, \lambda, k, m\rangle_{s_2} &= \hbar \lambda |j, \lambda, k, m\rangle_{s_2}, \\ \hat{M}_3 |j, \lambda, k, m\rangle_{s_2} &= \hbar m |j, \lambda, k, m\rangle_{s_2}, \\ \hat{K}_3 |j, \lambda, k, m\rangle_{s_2} &= \hbar k |j, \lambda, k, m\rangle_{s_2}, \end{aligned}$$

where numbers  $(j, \lambda, m, k)$  satisfy

$$\begin{aligned} 2\lambda, 2j, j + \lambda + m, j - \lambda + k &\in \mathbb{Z}, \\ j, j - \lambda, j + \lambda - |m|, j - \lambda - |k| &\geq 0. \end{aligned}$$

To make sure that  $|j, \lambda, k, m\rangle_{s_2} = |n\rangle$  we should further require that

$$\begin{aligned} j &= \frac{1}{4}(n_1 + n_2 + n_3 + n_4) \\ \lambda &= \frac{1}{4}(n_1 + n_2 - n_3 - n_4) \\ m &= \frac{1}{2}(n_1 - n_2) \\ k &= \frac{1}{2}(n_3 - n_4) \end{aligned}$$

or, equivalently,

$$\begin{aligned} n_1 &= j + \lambda + m \\ n_2 &= j + \lambda - m \\ n_3 &= j - \lambda + k \\ n_4 &= j - \lambda - k. \end{aligned}$$

Applying the standard procedure (sec. 2.1.1), we define oscillator coherent states  $|z\rangle_{oc}$  relative to  $z \in \mathbb{C}^4$  as joint eigenstates of the annihilation operators

$$\hat{a}_i |z\rangle_{oc} = z_i |z\rangle_{oc} \quad \text{for } i = 1, \dots, 4.$$

Using short-hand notation

$$|z|^2 = \sum_{i=1}^4 |z_i|^2, \quad z^n = \prod_{i=1}^4 z_i^{n_i}, \quad \text{and} \quad n! = \prod_{i=1}^4 n_i!,$$

the explicit expression for  $|z\rangle_{oc}$  can be written as

$$|z\rangle_{oc} = e^{-\frac{1}{2}|z|^2} \sum_{n \in \mathbb{N}_0^4} \frac{z^n}{\sqrt{n!}} |n\rangle.$$

The novelty of the present work begins here. The expression above can be rewritten in terms of rotator basis functions  $|j, \lambda, k, m\rangle_{s_2}$

$$|z\rangle_{oc} = \mathcal{N}(\alpha) \sum_{j, \lambda, m, k} \frac{\alpha_1^j \alpha_2^{j+m} \alpha_3^{j+k} \alpha_4^\lambda}{\sqrt{P(j, \lambda, k, m)}} |j, \lambda, k, m\rangle_{s_2},$$

with

$$\begin{aligned} P(j, \lambda, k, m) &= (j + \lambda + m)!(j + \lambda - m)!(j - \lambda + k)!(j - \lambda - k)! = n!, \\ \alpha &= (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (z_2^2 z_4^2, z_1 z_2^{-1}, z_3 z_4^{-1}, z_1 z_2 z_3^{-1} z_4^{-1}), \end{aligned}$$

and where  $\mathcal{N}(\alpha)$  is an appropriate normalization factor.

In order to recover the classical phase space of the rigid body, similarly to the Schwinger construction in sec. 2.3.3, we project  $|z\rangle_{\text{oc}}$  orthogonally onto the subspace  $\mathcal{H}_0$  spanned by functions with  $\lambda = 0$  and  $j \in \mathbb{N}_0$ . The parameters  $\alpha$  are chosen so that the resulting projected state no longer depends on  $\alpha_4$  and we come to the expression

$$|\alpha\rangle_{\text{cc}} = \mathcal{N}(\alpha) \sum_{j,m,k} \frac{\alpha_1^j}{(2j)!} \sqrt{\binom{2j}{j+m}} \alpha_2^{j+m} \sqrt{\binom{2j}{j+k}} \alpha_3^{j+k} |j, 0, m, k\rangle_{s_2}$$

parameterized now by  $\alpha \in \mathbb{C}^3$ , and where  $\mathcal{N}(\alpha)$  is again an appropriate normalization factor (the same as before provided that we set  $\alpha_4 = 0$ ).

For given  $j$  we recognize in the expression of  $|\alpha\rangle_{\text{cc}}$  the product of two Bloch coherent states of spin  $j$  (in the complex parameterization of the sphere that we have seen before in sec. 2.3.2). Specifically, with an appropriate re-parametrization that introduces  $\beta \in \mathbb{C}^3$  as function of  $\alpha$ ,

$$\beta = (\beta_1, \beta_2, \beta_3) = ((1 + |\alpha_2|^2)(1 + |\alpha_3|^2)\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3) \quad (21)$$

and assuming that the normalization factor is readjusted to fit the new parametrization, we have

$$|\beta\rangle_{\text{cc}} = \mathcal{N}(\beta) \sum_{j,m,k} \frac{\bar{\beta}_1^j}{(2j)!} F(j, m, \beta_2) F(j, k, \beta_3) |j, 0, m, k\rangle_{s_2}$$

with

$$F(j, n, \beta) = \frac{1}{(1 + |\beta|^2)^j} \sqrt{\binom{2j}{j+n}} \bar{\beta}^{j+n}$$

where we can identify  $|j, 0, m, k\rangle_{s_2}$  with  $|j, m\rangle_s |j, k\rangle_s$  and therefore obtain

$$|\beta\rangle_{\text{cc}} = \mathcal{N}(\beta) \sum_{j=0}^{\infty} \frac{\bar{\beta}_1^j}{(2j)!} |j, \beta_2\rangle_{\text{bc}} |j, \bar{\beta}_3\rangle_{\text{bc}}. \quad (22)$$

Comparing to (8) we see that one of the two Bloch states in each product is centered in  $\beta_2$  while the other in  $\bar{\beta}_3$ . This happens because the first state represents angular momentum in the laboratory frame, while the other represents the same angular momentum in the body frame where the commutation relations have anomalous sign.

We want now to determine explicitly the normalization factor  $\mathcal{N}(\beta)$ . Using the fact that Bloch states are already normalized, we have

$$\mathcal{N}(\beta)^{-2} = \sum_{j=0}^{\infty} \frac{|\beta_1|^{2j}}{(2j)!^2} = \frac{1}{2} (\text{J}_0(2|\beta_1|^{1/2}) + \text{I}_0(2|\beta_1|^{1/2}))$$

where

$$\text{J}_0(x) = \sum_{m=0}^{+\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{x}{2}\right)^{2m}$$

is a Bessel function of first kind and  $\text{I}_0(x) = \text{J}_0(ix)$  is a modified Bessel function of the first kind. We conclude that

$$|\beta\rangle_{\text{cc}} = \sqrt{\frac{2}{\text{J}_0(2|\beta_1|^{1/2}) + \text{I}_0(2|\beta_1|^{1/2})}} \sum_{j=0}^{\infty} \frac{\bar{\beta}_1^j}{(2j)!} |j, \beta_2\rangle_{\text{bc}} |j, \bar{\beta}_3\rangle_{\text{bc}}.$$

The weights  $\mathcal{N}(\beta)^2 |\beta_1|^{2j} (2j)!^{-2}$  in  $|\beta\rangle_{\text{cc}}$  are depicted in figure 2 for different values of the classical action  $J(|\beta_1|) \approx \hbar \sqrt{|\beta_1|/2}$  (see the discussion in the next section).

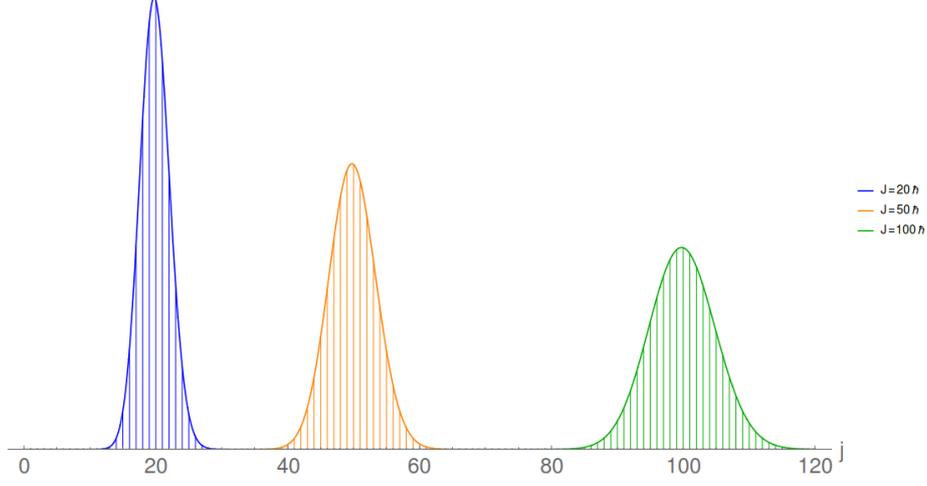


Figure 2: Weight distributions for different values of classical target action  $J$  ( $J = 20\hbar, 50\hbar, 100\hbar$ ) over  $j$ .

### 3.1 The classical choice of coherent state parameters

Like in section 2.3.4 we would also want to determine a positive measure density  $\mu$  on  $\mathbb{C}^3$  which defines a resolution of the identity for our set of coherent states, that is

$$\int_{\mathbb{C}^3} |\beta\rangle \langle \beta| \mu(\beta) d^6\beta = \sum_{j,m,k} |j, 0, m, k\rangle \langle j, 0, m, k|. \quad (23)$$

Here and in what follows we drop the subscript  $cc$ , since we have determined the set of coherent states we want to work with. Assuming that the measure is invariant under rotations, it is natural to require for  $\mu$  to take the form

$$\mu(\beta) = F(|\beta_1|)(1 + |\beta_2|^2)^{-2}(1 + |\beta_3|^2)^{-2}$$

for some non negative function  $F$ . In particular it is not restrictive introduce the function  $W$  defined as

$$W(x) = \frac{2F(x^{1/2})}{J_0(2x^{1/4}) + I_0(2x^{1/4})}$$

and rewrite the measure  $\mu$  in the following way

$$\mu(\beta) = \frac{1}{2}(J_0(2|\beta_1|^{1/2}) + I_0(2|\beta_1|^{1/2}))W(|\beta_1|^2)(1 + |\beta_2|^2)^{-2}(1 + |\beta_3|^2)^{-2}$$

with  $W$  non negative function to be determined. From the expression of the resolution of the identity for Bloch coherent states (9) parametrized by  $\beta_2$  and  $\beta_3$ , using the polar form for  $\beta_1$ , and integrating over its argument, we end up with

$$\int_0^{+\infty} W(x) x^j dx = \frac{((2j+1)!)^2}{\pi^3} \quad \text{where } j \in \mathbb{N}_0.$$

We recognize a Stieltjes moments problem for function  $W$ . If we define  $\rho(s)$  as

$$\rho(s) = \frac{1}{\pi^3} \Gamma(2s + 2)^2 \quad \text{where } s \in \mathbb{C}$$

we have that any solution  $W$  for the Stieltjes moments problem satisfies

$$\int_0^{+\infty} W(x) x^j dx = \rho(j) \quad \text{where } j \in \mathbb{N}_0.$$

Using the formula (see for instance eq. 10.43.19 of [30, 31])

$$\int_0^{\infty} t^{\mu-1} K_{\nu}(t) dt = 2^{\mu-1} \Gamma\left(\frac{\mu-\nu}{2}\right) \Gamma\left(\frac{\mu+\nu}{2}\right) \quad \text{where } |\operatorname{Re} \nu| < \operatorname{Re} \mu$$

for the modified Bessel function of the second kind  $K_{\nu}(t)$ , we have that

$$\int_0^{\infty} x^s \left(\frac{1}{\pi^3} K_0(2x^{1/4})\right) dx = \rho(s).$$

Consequently, if we set

$$W(x) = \frac{1}{\pi^3} K_0(2x^{1/4})$$

we can verify that  $W$  is positive and that it defines a solution of the moments problem (for some interesting insights into the relationship between the Stieltjes moments problem for  $W$  and the expression of  $\rho$  in a broader context see [32]). We conclude that the measure density that defines the resolution of identity for our set of coherent states (23) is given by

$$\mu(\beta) = \frac{1}{2\pi^3} (J_0(2|\beta_1|^{1/2}) + I_0(2|\beta_1|^{1/2})) K_0(2|\beta_1|^{1/2}) (1 + |\beta_2|^2)^{-2} (1 + |\beta_3|^2)^{-2}.$$

As done in sec. 2.3.4, we relate the complex parameters  $\beta$  to the classical phase space. We require that  $\beta_2$  and  $\beta_3$  parametrize each a sphere of radius  $J$ :  $\beta_2$  characterizes the angular momentum in the laboratory frame, while  $\beta_3$  characterizes the angular momentum in the body frame. We also require that  $\beta_1$  parametrizes the magnitude of the total angular momentum  $J$  and its conjugated angle  $\phi_J$ . Imposing the ansatz  $\beta_1 = b(J) e^{i\phi_J}$ , with  $b$  positive and increasing function, we have that

$$\mu(\beta) d^6\beta = \hbar^3 \frac{b(J)}{4J^2} (J_0(2b(J)^{1/2}) + I_0(2b(J)^{1/2})) K_0(2b(J)^{1/2}) \frac{db(J)}{dJ} \left(\frac{\omega}{2\pi\hbar}\right)^3.$$

Imposing the condition

$$\mu(\beta) d^6\beta = \left(\frac{\omega}{2\pi\hbar}\right)^3$$

we obtain

$$\frac{dJ(b)}{db} J^2(b) = \hbar^3 \frac{b}{4} (J_0(2b^{1/2}) + I_0(2b^{1/2})) K_0(2b^{1/2})$$

where  $J = J(b)$  can be seen as the inverse of  $b(J)$ . If we introduce the quantities  $y = J^3$  and  $x = 2b^{1/2}$ , and consider  $y = y(x)$  as a function of  $x$ , we can rewrite the previous equation as

$$\frac{dy(x)}{dx} = \frac{3\hbar^3}{32} x^3 (J_0(x) + I_0(x)) K_0(x).$$

Exploiting the fact that  $y(0) = 0$  this gives

$$\begin{aligned} y(x) &= \frac{3\hbar^3}{32} \int_0^x t^3 (J_0(t) + I_0(t)) K_0(t) dt \\ &= \frac{3\hbar^3}{128} \pi^{-1/2} \left( 16G_{0,4}^{3,0} \left( 1, 1, 3/2, 0 \mid \frac{x^4}{64} \right) + G_{2,4}^{2,2} \left( \frac{1, 5/2}{2, 2, 0, 2} \mid x^2 \right) \right) \end{aligned}$$

where

$$G_{p,q}^{m,n} \left( \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \mid z \right)$$

denotes a Meijer G-function (see for instance [33, 34]). Going back to the original parameters this results in

$$\begin{aligned} J(b) &= \hbar \frac{3^{1/3}}{4 \cdot 2^{1/3} \pi^{1/6}} \left( 16G_{0,4}^{3,0} \left( 1, 1, 3/2, 0 \mid \frac{b^2}{4} \right) + G_{2,4}^{2,2} \left( \frac{1, 5/2}{2, 2, 0, 2} \mid 4b \right) \right)^{1/3} \\ &= \frac{\hbar\sqrt{b}}{2} \left( \frac{3}{2\sqrt{\pi}} \left( \frac{1}{\sqrt{2}} G_{0,4}^{3,0} \left( \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, -\frac{3}{4} \mid \frac{b^2}{4} \right) + G_{2,4}^{2,2} \left( \frac{-\frac{1}{2}, 1}{\frac{1}{2}, \frac{1}{2}, -\frac{3}{2}, \frac{1}{2}} \mid 4b \right) \right) \right)^{1/3}. \end{aligned} \quad (24)$$

Despite the complex expression it is possible to analyze the asymptotic behavior for  $b \sim \infty$  and to obtain:

$$J(b) \sim \frac{\hbar\sqrt{b}}{2} = J_{\text{as}}(b).$$

We can at this point check the expectation values of quantum observables on coherent states. Using the formulas for the expectation values of Bloch coherent states we have

$$\begin{aligned} \langle \beta | \hat{I} | \beta \rangle &= \frac{2\hbar}{J_0(2|\beta_1|^{1/2}) + I_0(2|\beta_1|^{1/2})} \sum_{j=0}^{\infty} \frac{j|\beta_1|^{2j}}{((2j)!)^2} \\ &= \frac{\hbar\sqrt{|\beta_1|} I_1(2|\beta_1|^{1/2}) - J_1(2|\beta_1|^{1/2})}{2 J_0(2|\beta_1|^{1/2}) + I_0(2|\beta_1|^{1/2})} \\ &\xrightarrow{|\beta_1| \sim \infty} \frac{\hbar\sqrt{|\beta_1|}}{2} - \frac{\hbar}{8} = J_{\text{as}}(|\beta_1|) - \frac{\hbar}{8}. \end{aligned}$$

Note that the sum in the above equation can be evaluated using the Taylor expansion of  $J_0$  and its derivative. Similar expressions are easily obtained for  $\langle \beta | \hat{M}_i | \beta \rangle$  and for  $\langle \beta | \hat{K}_i | \beta \rangle$ . In the limit of large  $|\beta_1|$ , which is equivalent to small  $\hbar$ , and so it can be thought as a semi-classical limit, we come to the expected agreement (up to a small term of order of  $\hbar$ ):

$$\begin{aligned} \langle \beta | \hat{M}_i | \beta \rangle &\sim \left( J(|\beta_1|) - \frac{\hbar}{8} \right) p_i(\beta_2) \\ \langle \beta | \hat{K}_i | \beta \rangle &\sim \left( J(|\beta_1|) - \frac{\hbar}{8} \right) p_i(\beta_3) \\ \langle \beta | \hat{I} | \beta \rangle &\sim J(|\beta_1|) - \frac{\hbar}{8} \end{aligned}$$

where  $p_i(\beta_1)$  and  $p_i(\beta_2)$  represent the  $i^{\text{th}}$  components of the unit vectors obtained from  $\beta_1$  and  $\beta_2$  respectively through the inverse of the complex stereographical projection (10) (with a choice of  $\rho$  equal to 1). The comparison of  $\langle \beta | \hat{I} | \beta \rangle$  and  $J(|\beta_1|)$  is presented in figure 3.

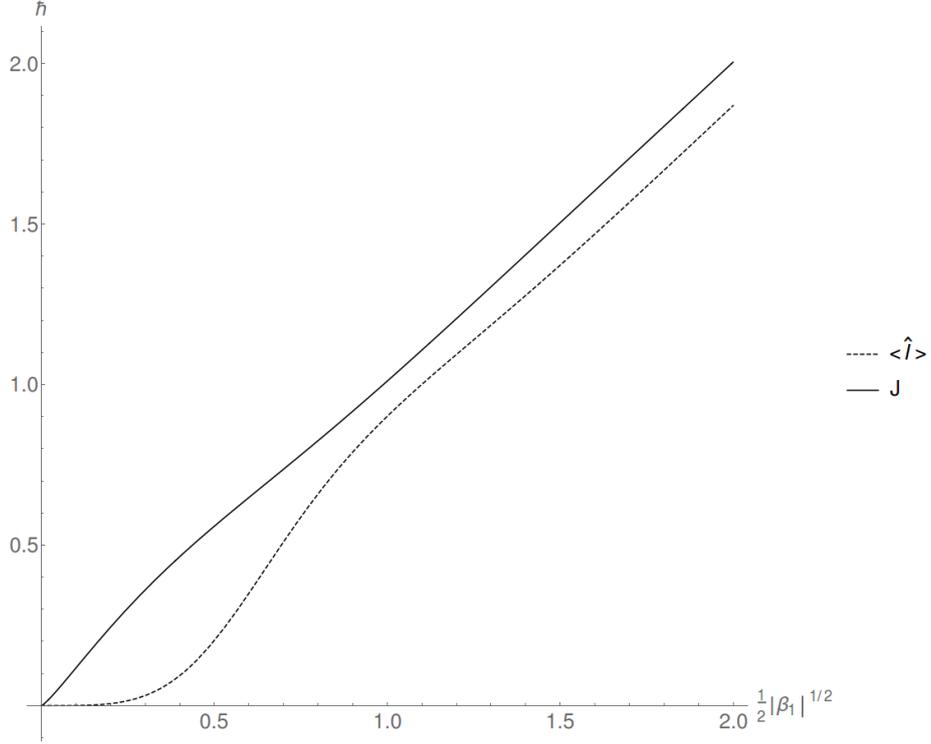


Figure 3: Comparison between the expectation value of  $\hat{I}$  as defined in equation (19) and  $J(|\beta_1|)$  as defined in equation (24) for coherent states on  $T^*\text{SO}(3)$  as functions of complex parameter  $\beta_1$ .

### 3.2 Body standard coordinates: trivialization of $T^*\text{SO}(3)$

To draw a more direct relationship to the classical system we can reparameterize the family of coherent states employing a more commonly used set of parameters. We observe that, for fixed  $\beta \in \mathbb{C}^3$ , there exists a unique  $g_1 \in \text{SO}(3)$  such that

$$U^{(j)}(g_1)|j, \beta_3\rangle_{\text{bc}} = |j, \beta_2\rangle_{\text{bc}} \quad \forall j \in \mathbb{N}_0$$

where  $U^{(j)}(g)$  is the unitary representative of  $g$  in the irreducible representation of  $\text{SO}(3)$  of weight  $j$ . Moreover, there exists a unique  $g_2 \in \text{SO}(3)$ , which leaves  $\beta_2$  invariant and changes the phase of the Bloch function so that

$$U^{(j)}(g_2)|j, \beta_2\rangle_{\text{bc}} = \exp(-ij\phi_J)|j, \beta_2\rangle_{\text{bc}} \quad \forall j \in \mathbb{N}_0$$

(recall that  $\phi_J$  is the argument of  $\bar{\beta}_1$ ). Consequently we have that there exists a unique  $g = g_2 \circ g_1 \in \text{SO}(3)$  such that

$$|\beta\rangle = \mathcal{N}(b(J)) \sum_{j=0}^{\infty} \frac{b(J)^j}{(2j)!} U^{(j)}(g)|j, \beta_3\rangle_{\text{bc}} |j, \bar{\beta}_3\rangle_{\text{bc}}$$

(recall that  $b(J)$  is the absolute value of  $\beta_1$ ). We can think about  $g$  as being the classical rotation that corresponds to the coherent state  $|\beta\rangle$  and that relates the laboratory and the

body frames. If to any  $v \in \mathbb{R}^3$  we associate  $\theta(v) \in \mathbb{C}$  such that

$$\theta(v) = \frac{v_1 + iv_2}{\|v\| - v_3},$$

we can relabel, our coherent state  $|\beta\rangle$  using  $k_c \in \mathbb{R}^3$  and  $g \in \text{SO}(3)$

$$|g, k_c\rangle_{\text{BF}} = \mathcal{N}(b(\|k_c\|)) \sum_{j=0}^{\infty} \frac{b(\|k_c\|)^j}{(2j)!} U^{(j)}(g) |j, \theta(k_c)\rangle_{\text{bc}} |j, \overline{\theta(k_c)}\rangle_{\text{bc}}$$

or equivalently, using  $m_c \in \mathbb{R}^3$  (the equivalence being given by  $m_c = g k_c$ ),

$$|g, m_c\rangle_{\text{LF}} = \mathcal{N}(b(\|m_c\|)) \sum_{j=0}^{\infty} \frac{b(\|m_c\|)^j}{(2j)!} |j, \theta(m_c)\rangle_{\text{bc}} \left( U^{(j)}(g) \right)^T |j, \overline{\theta(m_c)}\rangle_{\text{bc}}.$$

In this way we identify  $\text{T}^*\text{SO}(3)$  with  $\text{SO}(3) \times \text{so}(3)^*$ , where  $\text{so}(3)^*$  is in turn identified with  $\mathbb{R}^3$ , through left and right trivializations. The respective parameters  $(g, k_c)$  or  $(g, m_c)$  parametrize the families of coherent states  $|g, k_c\rangle_{\text{BF}}$  and  $|g, m_c\rangle_{\text{LF}}$  giving representations in the body or in the laboratory frame respectively. In applications this means that we can specify the orientation of  $\mathbf{J}$  in the laboratory frame or in the body frame by vectors  $m_c$  and  $k_c$  respectively, relate the two frames by the rotation  $g$ , and use coherent states  $|g, m_c\rangle_{\text{LF}}$  or  $|g, k_c\rangle_{\text{BF}}$  to follow the evolution of the system in terms of  $g$  and  $m_c$  or  $k_c$ .

## 4 Coherent states on $\text{T}^*\text{SO}(3)$ in physical applications

We conclude the paper with a practical application of our coherent states. Considering a simple, yet relevant, example of the quantum spherical top, we analyze the evolution of a given coherent state. We use the Husimi distribution which allows representing any quantum state as a probability distribution on the classical phase space. We use this example to verify our construction of  $\text{T}^*\text{SO}(3)$  coherent states. In some sense, they behave as one might expect: they stay centered on the underlying classical trajectory while exhibiting diffusion.

It is important, however, to note that there is an evolution at all. In fact, since the norm of the angular momentum  $\mathbf{J}$  is conserved, it is common to study quantum rotors for a fixed  $j$  (thus using Bloch coherent states on  $\mathbb{S}^2$ ). For the spherical top, the unitary operator that expresses the dynamics at fixed  $j$  is trivial, and is reduced to a phase change which is not physically meaningful. This can be interpreted as the quantum equivalent of the fact that the classical *reduced* Euler's equations of motion for the spherical rigid body on the *reduced* phase space  $\mathbb{S}^2$  (spanned by the orientations of  $\mathbf{J}$  with respect to the body frame) are trivial. On the other hand, on the complete phase space, the classical dynamics of this body is seen as rotation in the laboratory frame about vector  $\mathbf{J}$  with constant frequency proportional to  $\|\mathbf{J}\|$ .

We turn to numerical simulations in order to demonstrate how this classical dynamics is reflected in the quantum system. Considering the semi-classical regime with  $\hbar$  small compared to  $\|\mathbf{J}\|$ , we will see that the Husimi distribution of our coherent states is localized for long times near the respective classical orbits. This justifies most clearly the choice of our family of coherent states.

## 4.1 Bargmann and Husimi representations

We recall the basics of the Husimi distribution [35] which we will use in sec. 4.2 to represent our coherent states as a probability distribution on  $T^*\text{SO}(3)$ . Consider a symplectic manifold  $\mathcal{M}$  with points  $p \in \mathcal{M}$ , a quantum Hilbert space  $\mathcal{H}$ , and a family of coherent states  $|p\rangle \in \mathcal{H}$  which is associated to  $\mathcal{M}$ . For any (normalized) state  $|\psi\rangle \in \mathcal{H}$ , we can construct its Bargmann representation

$$B : \mathcal{H} \times \mathcal{M} \rightarrow \mathbb{C} : (|\psi\rangle, p) \mapsto B_{|\psi\rangle}(p) := \langle p|\psi\rangle.$$

The complex valued function  $B_{|\psi\rangle}$  determines fully the state  $|\psi\rangle$ : using the measure  $d\mu$  on  $\mathcal{M}$  that defines the resolution of identity for the family  $|p\rangle$  we have

$$\int_{\mathcal{M}} B_{|\psi\rangle}(p)|p\rangle d\mu(p) = \int_{\mathcal{M}} |p\rangle \langle p| d\mu(p)|\psi\rangle = \mathbf{1}|\psi\rangle = |\psi\rangle.$$

Closely related to the Bargmann representation of state  $|\psi\rangle$  is its Husimi distribution

$$H_{|\psi\rangle}(p) = |B_{|\psi\rangle}(p)|^2 = \langle \psi|p\rangle \langle p|\psi\rangle.$$

Function  $H_{|\psi\rangle}$  is real and its range equals  $[0, 1]$  for any normalized  $|\psi\rangle$ . Furthermore,

$$\int_{\mathcal{M}} H_{|\psi\rangle}(p) d\mu(p) = \langle \psi| \int_{\mathcal{M}} |p\rangle \langle p| d\mu(p) |\psi\rangle = \langle \psi|\mathbf{1}|\psi\rangle = \langle \psi|\psi\rangle = 1.$$

In particular this defines a probability measure  $d\mu_{|\psi\rangle} = H_{|\psi\rangle} d\mu$  on  $\mathcal{M}$ .

It is possible to see, as an example, that the Husimi distribution of an Euclidean coherent state  $|z^*\rangle_{\text{oc}}$  with  $\mathcal{M} \sim \mathbb{C}$  (see sec. 2.1) is a Gaussian centered at  $z^*$

$$H_{|z^*\rangle_{\text{oc}}}(z) = |\text{oc}\langle z|z^*\rangle_{\text{oc}}|^2 = \exp(-|z - z^*|^2),$$

while in the case of Bloch coherent states (sec. 2.2),

$$H_{|j, z^*\rangle_{\text{bc}}}(z) = |\text{bc}\langle j, z|j, z^*\rangle_{\text{bc}}|^2 = \cos^{2j}(\angle z z^*)$$

defines through (10) a similar probability distribution on the Riemann sphere  $\mathbb{S}_{\rho(j)}^2$ .

Considering coherent states  $|p\rangle$  as states that are the ‘‘most localized’’ at point  $p$  in the classical phase space  $\mathcal{M}$ , the values of  $H_{|\psi\rangle}$  can be interpreted as the probability of finding the state  $|\psi\rangle$  at  $p \in \mathcal{M}$ . Of a particular interest to our work is the study of the Husimi distribution of the evolution of a coherent state  $|p\rangle$ , a time-dependent probability distribution on  $\mathcal{M}$ , under a given quantum Hamiltonian, and the comparison of this evolution to the corresponding classical orbit passing through  $p$ .

## 4.2 The spherical top

As a concrete application, let us consider the quantum spherical top. With an appropriate time rescaling, the Hamiltonian of this system can always be put in the simple form

$$\hat{H} = \frac{1}{2} \hat{K}^2,$$

where  $\hat{K}$  represents angular momentum  $\hat{J}$  in the body frame, see eqs. (19b). Note that the same Hamiltonian is used in [19] to evolve coherent states on  $T^*\mathbb{S}^2$ , so it is instructive to compare what we do below with the computations in sec. 4 of [19].

Recall that the eigenspectrum of  $\hat{H}$  is discrete and equals  $\frac{1}{2}\hbar^2 j(j+1)$  with  $j \in \mathbb{N}_0$ , and that these eigenvalues (energies) are  $(2j+1)^2$  degenerate. Under Hamiltonian  $\hat{H}$ , the evolution of the  $T^*\text{SO}(3)$  coherent state with parameter  $\beta \in \mathbb{C}^3$  is given by

$$|\beta; t\rangle = \exp\left(\frac{\hat{H}t}{i\hbar}\right)|\beta\rangle = \mathcal{N}(\beta_1) \sum_{j=0}^{+\infty} e^{i\omega_j t} \bar{\beta}_1^j |j, \beta_2\rangle_{sc} |j, \bar{\beta}_3\rangle_{sc}.$$

where the frequencies associated to the quasi-periodic dynamics are

$$\omega_j = -\frac{\hbar}{2}j(j+1).$$

Since  $j$  is integer (and so  $j(j+1)$  is even) we have that, if we define  $T = 2\pi\hbar^{-1}$ ,

$$T\omega_j \in 2\pi\mathbb{Z} \text{ for all } j \in \mathbb{N}_0, \quad (25)$$

it is also possible to see that  $T$  is the smallest positive number satisfying this condition (for instance by checking this on the frequency  $\omega_1$ ). An immediate consequence of expression (25) is

$$|\beta; t+T\rangle = |\beta; t\rangle \text{ for all } t \in \mathbb{R} \text{ and } \beta \in \mathbb{C}^3.$$

This allows us to conclude that the dynamics is periodic with minimal period  $T$  (independent from the orbit). This purely quantum phenomenon depends on  $\hbar$  and its origin lies in the resonant structure of the discrete spectrum of  $\hat{H}$  whose degenerate eigenvalues are separated by gaps that are integer multiples of  $\hbar^2$ . It has no relation to the periodicity of the classical motion for which the period depends on the initial conditions  $\beta$ . Excluding the trivial case where  $\beta_1 = 0$  (in this case  $|\beta\rangle$  is a stationary state), for a given coherent state  $|\beta\rangle$  the condition ensuring that  $|\beta; t\rangle$ , for some  $t$  such that  $0 < t < T$ , is coherent, is equivalent to the existence of  $\alpha \in \mathbb{C}$  such that

$$e^{i\omega_j t} = \alpha^j \text{ for all } j \in \mathbb{N}_0.$$

After some manipulations this condition can be rewritten in the form

$$\frac{t}{T} \in \mathbb{Z}$$

which has no solutions, henceforth we conclude that  $|\beta; t\rangle$  is not a coherent state.

Fixed initial condition  $\beta \in \mathbb{C}^3$ , further insight on the quantum/classical dynamics relationship can be provided by the Husimi distribution  $H_{|\beta; t\rangle}$  of  $|\beta; t\rangle$ . First of all, for visualization purpose, we need to reduce the number of coordinates in the phase space we will consider. Drawing the comparison to the classical case, we observe that only the angle  $\phi_J$  conjugate to  $\|\mathbf{J}\|$  evolves. Furthermore, noting that  $\phi_J$  is represented by the argument of  $\beta_1$ , we can restrict ourselves on the one-dimensional section of the phase space parametrized by  $\lambda \in [0, 2\pi)$ , and defined as

$$\lambda \rightarrow (e^{i\lambda}\beta_1, \beta_2, \beta_3).$$

On this section the Husimi distribution of  $|\beta; t\rangle$  becomes a function of  $\lambda$  and  $t$  alone. If we use (24) to define the corresponding *target value* of classical action  $J$  (the norm  $\|\mathbf{J}\|$  of the angular momentum), we have that the classical evolution of the respective conjugate angle is

$$\phi_J(t) = \phi_J(0) + tJ,$$

which in the  $(\lambda, t)$  plane is described by a line (we recall that, in our choice of parameterization of the section,  $\phi_J = \phi_J(0)$  corresponds to  $\lambda = 0$ )

$$\lambda^*(t) = tJ. \quad (26)$$

In fig. 4 we represent the Husimi distribution of  $|\beta; t\rangle$  on the  $(\lambda, t)$  plane along with the corresponding classical orbit (26) for different values of  $J$ . It can be well seen that this distribution is localized in  $\lambda$  and clearly follows the classical orbit before spreading. In other words, our coherent state follows (at least for some time) the evolution of the classical system. The localization time scale, the spatial localization, and the quantum-classical correspondence itself improve for large  $J$ .

## 5 Discussion and perspectives

We constructed in sec. 3 a new family of coherent quantum states that parametrizes the phase space of the rigid body. Exploiting the resolution of identity and the associated measure, we established the most direct and detailed relation between the quantum system and its classical analog. This relation is confirmed by the computation of expectation values of principal quantum observables, and is further supported numerically by how well the evolution of our coherent states replicates in the semi-classical limit the behavior of the classical system. Our construction makes an opening for further research and applications, some of which are suggested below.

Simple expressions in the spherical top case (sec. 4) make it possible to compute explicitly the  $\hbar \rightarrow 0$  limit for the quantum evolution. This can in turn give the correct classical dynamics and thus support further our construction. The quantum and classical dynamics of the symmetric top, while being more complex, can still be represented in closed form, and a similar analysis can follow.

More generally, since it is possible to express quantum evolution of the Bargmann distribution of a state as a set of state-independent partial differential equations (with boundary conditions determined by the state), it is interesting to verify how these equations can be reduced, to a factor of the order of  $\hbar$ , to the classical equations of motion for the densities on the phase space. In particular, this can be studied using our coherent states and the classical and quantum rigid body Hamiltonian.

The author of [8, 9] used geometric quantization to define coherent states on  $T^*G$  with  $G$  a compact Lie group. Since this should work for  $SO(3)$ , it is interesting to compare in detail the states in [8, 9] to our construction in sec. 3.

Our states open the way for the numerical analysis of more complicated systems, notably the study of the triaxial rigid and semi-rigid bodies, such as asymmetric top molecules. Particularly interesting are perturbations depending on the orientation of such bodies with respect to the laboratory frame. For example, we can study interactions of the external electric fields with molecules possessing permanent dipole moments, which can be modelled as heavy tops, see [36, 37, 38], which in turn can be either classically integrable (for instance in the case of the Lagrange or the Kovalevskaya tops) or, in the general non-symmetric case, non-integrable. While the necessary computations pose certain challenges, they can be highly rewarding. For instance, we can make possible a detailed analysis of the quantum counterpart of the classical dynamics near unstable relative equilibria, or in the presence of bifurcations of stable equilibria, such as gyroscopic destabilization (see [39] and references therein), and, generally, within the emerging chaotic regions in the phase space.

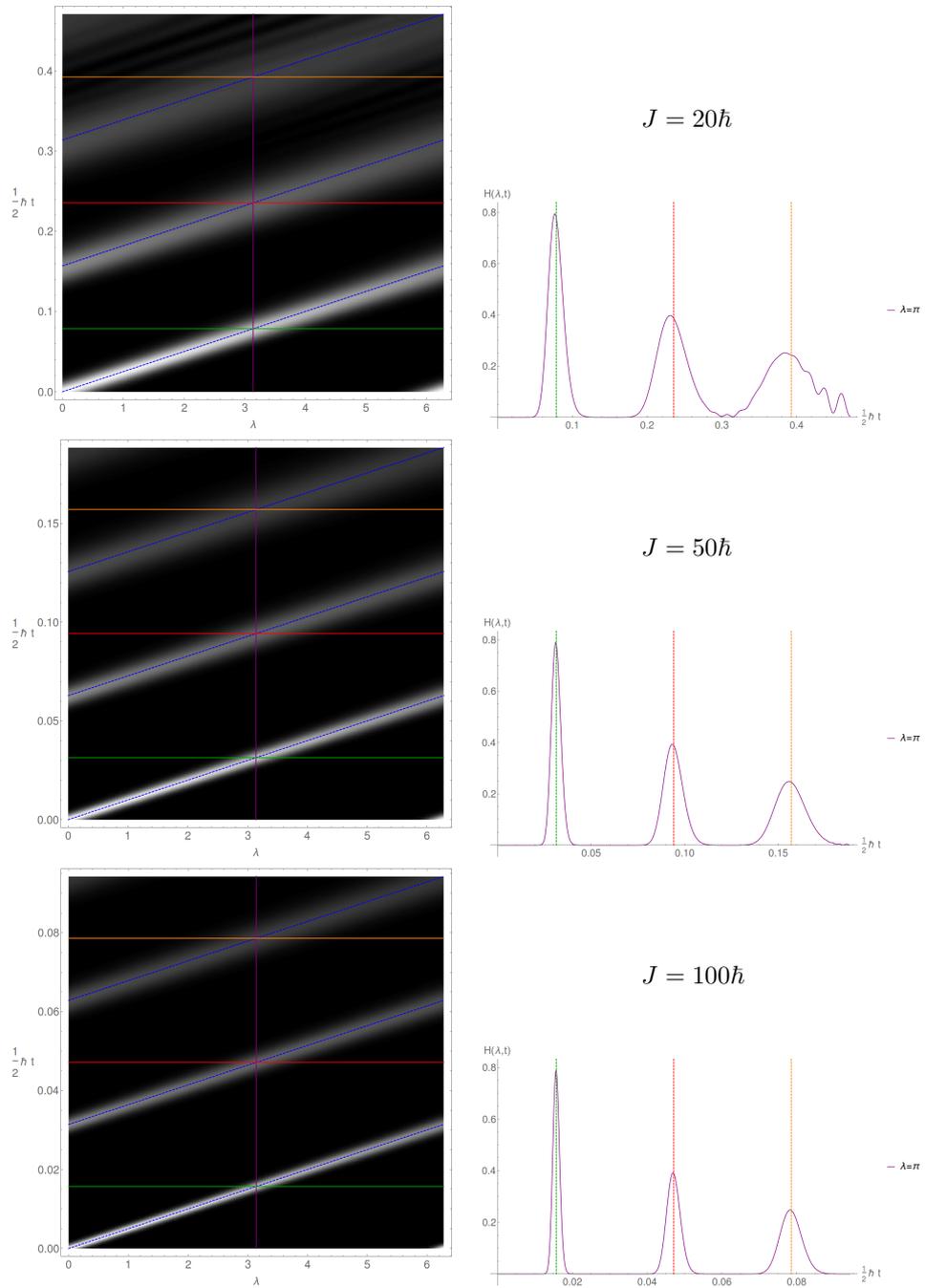


Figure 4: Quantum evolution of the canonical angle  $\lambda$  conjugate to the classical action  $J$  of the spherical top. Each row corresponds to one fixed classical target value of  $J\hbar^{-1}$  (respectively 20, 50 and 100); left column presents Husimi distribution of  $|\beta; t\rangle$  as function of  $\lambda$  and time; right column shows three constant successive  $\lambda = \pi \pmod{2\pi}$  sections of this distribution (first three successive returns).

It is possible and important to uncover more fully the relation of Schwinger-like coherent states to the parallel classical construction for the 2 and 4-mode isotropic oscillator systems and their derivatives. Specifically, reducing the 1:1 resonance symmetry  $\mathbb{S}^1$ , we come to the reduced Euler top described using the standard Bloch coherent states. In higher dimension, reduction of the 1:1:1:1 resonance symmetry, together with an additional  $\mathbb{S}^1$  symmetry, which is a classical analogue of the requirement for the eigenvalues  $\lambda$  of  $\hat{A}$  in (19a) to be equal 0, results in the description of the regularized Keplerian systems, most notably, the hydrogen atom and its perturbations [25], within the framework of the Kustanheimo-Stiefel approach. So coherent states, similar to the ones in sec. 3, but with unrestricted values of  $j$ , which becomes the principal (shell) quantum number  $n$ , can be used to analyze quantum-classical correspondence for these systems, and more generally—for systems on the phase space  $\mathbb{S}^2 \times \mathbb{S}^2$ , including bifurcations, integer and fractional monodromy, adiabatic invariants, and much more. In this context, it is interesting to relate our states to the  $SO(4)$  coherent states on  $\mathbb{S}^3$  [40, chapt. 9]. Further generalizations to non-zero  $\lambda$  or/and to resonances like 1:1:2:2 can be used for similar studies of the vibrational dynamics of linear polyatomic molecules such as  $C_2H_2$ ,  $C_2N_2$ , and others.

## Acknowledgments

This work was made possible by a one-year fellowship from Université Franco-Italienne, grant VINCI-2015 C4-54. It is part of the project developed together with Professors Boris Zhilinskiĭ and Gregory Ezra, and Doctor Arnaud Cuisset.

The authors would like to thanks the anonymous referee for his careful reading of the manuscript and his insightful comments and suggestions.

## References

- [1] Roy J Glauber. Coherent and incoherent states of the radiation field. *Phys. Rev.*, 131(6):2766–2788, 1963.
- [2] E. C. G. Sudarshan. Equivalence of semiclassical and quantum mechanical descriptions of statistical light beams. *Phys. Rev. Lett.*, 10(7):277–279, 1963.
- [3] John R Klauder. The action option and a Feynman quantization of spinor fields in terms of ordinary  $C$ -numbers. *Ann. Phys. (N.Y.)*, 11(2):123–168, 1960.
- [4] John R. Klauder and Bo-Sture Skagerstam. *Coherent States: Applications in Physics and Mathematical Physics*. World Scientific, 1985.
- [5] R. J. Glauber. *Quantum Theory of Optical Coherence. Selected Papers and Lectures*. Wiley-VCH, Weinheim, 2007.
- [6] Wei-Min Zhang, Da Hsuan Feng, and Robert Gilmore. Coherent states: theory and some applications. *Rev. Mod. Phys.*, 62(4):867–927, 1990.
- [7] Askold Perelomov. *Generalized coherent states and their applications*. Springer Science & Business Media, 2012.
- [8] Brian C Hall. Geometric quantization and the generalized segal–bargmann transform for lie groups of compact type. *Communications in mathematical physics*, 226(2):233–268, 2002.

- [9] Brian C Hall. The Segal-Bargmann coherent state transform for compact Lie groups. *J. Func. Anal.*, 122(1):103–151, 1994.
- [10] Nicholas Michael John Woodhouse. *Geometric quantization*. Oxford University Press, 1997.
- [11] R. Gilmore. Geometry of symmetrized states. *Ann. Phys. (N.Y.)*, 74:391–463, 1972.
- [12] A. M. Perelomov. Coherent states for arbitrary Lie group. *Commun. Math. Phys.*, 26(3):222–236, 1972.
- [13] S. A. Deshpande and G. S. Ezra. Quantum state reconstruction for rigid rotors. *Chem. Phys. Lett.*, 440:341–347, 2007.
- [14] Dmitry V Zhdanov and Tamar Seideman. Wigner representation of the rotational dynamics of rigid tops. *Physical Review A*, 92(1):012129, 2015.
- [15] Lars Christiansen, Jens H. Nielsen, Lauge Christensen, Benjamin Shepperson, Dominik Pentlehner, and Henrik Stapelfeldt. Laser-induced Coulomb explosion of 1,4-diidobenzene molecules: Studies of isolated molecules and molecules in helium nanodroplets. *Phys. Rev. A*, 93:023411, Feb 2016.
- [16] Simon S Viftrup, Vinod Kumarappan, Sebastian Trippel, Henrik Stapelfeldt, Edward Hamilton, and Tamar Seideman. Holding and spinning molecules in space. *Physical review letters*, 99(14):143602, 2007.
- [17] Maxim Artamonov and Tamar Seideman. Theory of three-dimensional alignment by intense laser pulses. *The Journal of chemical physics*, 128(15):154313, 2008.
- [18] F. T. Arecchi, E. Courtens, R. Gilmore, and H. Thomas. Atomic coherent states in quantum optics. *Phys. Rev. A*, 6(6):2211–2237, 1972.
- [19] K Kowalski, J Rembieliński, and J Zawadzki. On the quantum dynamics of the rigid rotor. *Journal of Physics A: Mathematical and Theoretical*, 48(3):035301, 2014.
- [20] P Gulshani. Oscillator-like coherent states of an asymmetric top. *Phys. Lett. A*, 71(1):13–16, 1979.
- [21] P Gulshani. Generalized Schwinger boson realizations and the oscillator-like coherent states of the rotation groups and the asymmetric top. *Canad. J. Phys.*, 57(7):998–1021, 1979.
- [22] Michèle Irac-Astaud. Molecular-coherent-states and molecular-fundamental-states. *Rev. Math. Phys.*, 13(11):1437–1457, 2001.
- [23] Julian Schwinger. On angular momentum. In L. C. Biedenharn and H. Van Dam, editors, *Quantum theory of angular momentum*, pages 229–279. Academic Press, New York, 1965.
- [24] P. W. Atkins and J. C. Dobson. Angular momentum coherent states. *Proc. Roy. Soc. London. Ser. A*, 321(1546):321–340, 1971.
- [25] K. Efstathiou and D. A. Sadovskii. Normalization and global analysis of perturbations of the hydrogen atom. *Rev. Mod. Phys.*, 82(3):2099–2154, Aug 2010.

- [26] J. M. Radcliffe. Some properties of coherent spin states. *J. Phys. A*, 4:313–323, 1971.
- [27] R. H. Cushman and D. L. Rod. Reduction of the semisimple 1:1 resonance. *Physica D*, 6:105–112, 1982.
- [28] R. H. Cushman and L. Bates. *Global aspects of classical integrable systems*. Birkhauser, Basel, 1997.
- [29] K. Efsthathiou and D. A. Sadovskii. *Geometric Mechanics and Symmetry. The Peyresq Lectures*, chapter No Polar Coordinates. Lectures by Richard Cushman, pages 211–302. Cambridge University Press, Cambridge, UK, Jun 2005.
- [30] NIST Digital Library of Mathematical Functions. <http://dlmf.nist.gov/>, Release 1.0.11 of 2016-06-08. Online companion to [31].
- [31] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, editors. *NIST Handbook of Mathematical Functions*. Cambridge University Press, New York, NY, 2010. Print companion to [30].
- [32] Karol A Penson, Pawel Blasiak, Gerard H E Duchamp, Andrzej Horzela, and Allan I Solomon. On certain non-unique solutions of the Stieltjes moment problem. *Computer Science*, 12(2):295–306, 2009.
- [33] Yudell L. Luke. *The Special Functions and their Approximations, Vol. 1*. Academic Press, New York, 1969.
- [34] Yudell L. Luke. *Mathematical Functions and their Approximations*. Academic Press Inc., New York, 1975.
- [35] Kôdi Husimi. Some formal properties of the density matrix. *Nippon Sugaku-Buturigakkwai Kizi Dai 3 Ki*, 22(4):264–314, 1940.
- [36] I. N. Kozin and R. M. Roberts. Monodromy in the spectrum of a rigid symmetric top molecule in an electric field. *J. Chem. Phys.*, 118(23):10523–10533, 2003.
- [37] C. A. Arango, W. W. Kennerly, and G. S. Ezra. Quantum and classical mechanics of diatomic molecules in tilted fields. *J. Chem. Phys.*, 122:184303/1–15, 2005.
- [38] C. A. Arango and G. S. Ezra. Classical mechanics of dipolar asymmetric top molecules in collinear static electric and nonresonant linearly polarized laser fields: energy-momentum diagrams, bifurcations and accessible configuration space. *Int. J. Bifurc. Chaos*, 18:1127–1149, 2008.
- [39] Arnaud Cuisset, Olivier Pirali, and Dmitrii A. Sadovskii. Gyroscopic destabilization of molecular rotation and quantum bifurcation observed in the structure of the  $\nu_{23}$  fundamental of dimethylsulfoxide. *Phys. Rev. Lett.*, 109:094101, Aug 2012.
- [40] Monique Combescure and Didier Robert. *Coherent states and applications in mathematical physics*. Springer Science & Business Media, 2012.