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# ON THE GOTZMANN THRESHOLD OF MONOMIALS

V. BONANZINGA AND S. ELIAHOU

ABSTRACT. Let  $R_n = K[x_1, \dots, x_n]$  be the  $n$ -variable polynomial ring over a field  $K$ . Let  $S_n$  denote the set of monomials in  $R_n$ . A monomial  $u \in S_n$  is a *Gotzmann monomial* if the Borel-stable monomial ideal  $\langle u \rangle$  it generates in  $R_n$  is a Gotzmann ideal. A long-standing open problem is to determine all Gotzmann monomials in  $R_n$ . Given  $u_0 \in S_{n-1}$ , its *Gotzmann threshold* is the unique nonnegative integer  $t_0 = \tau_n(u_0)$  such that  $u_0 x_n^t$  is a Gotzmann monomial in  $R_n$  if and only if  $t \geq t_0$ . Currently, the function  $\tau_n$  is exactly known for  $n \leq 4$  only. We present here an efficient procedure to determine  $\tau_n(u_0)$  for all  $n$  and all  $u_0 \in S_{n-1}$ . As an application, in the critical case  $u_0 = x_2^d$ , we determine  $\tau_5(x_2^d)$  for all  $d$  and we conjecture that for  $n \geq 6$ ,  $\tau_n(x_2^d)$  is a polynomial in  $d$  of degree  $2^{n-2}$  and dominant term equal to that of the  $(n-2)$ -iterated binomial coefficient

$$\left( \binom{d}{2} \right).$$

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## 1. INTRODUCTION

Let  $K$  be a field. Let  $R_n = K[x_1, \dots, x_n]$  be the  $n$ -variable polynomial algebra over  $K$  endowed with the standard grading  $\deg(x_i) = 1$  for all  $i$ . We denote by  $S_n = \{x_1^{a_1} \cdots x_n^{a_n} \mid a_i \in \mathbb{N} \text{ for all } i\}$  the set of monomials in  $R_n$ , and by  $S_{n,d} \subset S_n$  the subset of monomials  $u$  of degree  $\deg(u) = \sum_i a_i = d$ .

A monomial ideal  $J \subseteq R_n$  is said to be *Borel-stable* or *strongly stable* if for any monomial  $v \in J$  and any variable  $x_j$  dividing  $v$ , one has  $x_i v / x_j \in J$  for all  $1 \leq i < j$ . Given a monomial  $u \in S_n$ , let  $\langle u \rangle$  denote the smallest Borel-stable monomial ideal in  $R_n$  containing  $u$ , and let

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$B(u)$  denote the unique minimal system of monomial generators of  $\langle u \rangle$ . Then  $B(u)$  is the smallest set of monomials containing  $u$  and stable under the operations  $v \mapsto vx_i/x_j$  whenever  $x_j$  divides  $v$  and  $i \leq j$ . See e.g. [6].

Recall that a homogeneous ideal  $I \subseteq R_n$  is a *Gotzmann ideal* if, from a certain degree on, its Hilbert function attains Macaulay's lower bound. See e.g. [4, 8] for more details. Determining which homogeneous ideals are Gotzmann ideals is notoriously difficult. See e.g. [2, 3, 9, 10, 11, 13, 14, 15, 16, 17, 18] for more on Gotzmann ideals. In this paper, we focus on this question for principal-Borel monomial ideals  $\langle u \rangle$ . Whence the following definition.

**Definition 1.1.** *Let  $u \in S_n$ . We say that  $u$  is a Gotzmann monomial if its associated Borel-stable monomial ideal  $\langle u \rangle$  is a Gotzmann ideal.*

Even for this restricted family of ideals  $\langle u \rangle$ , determining those which are Gotzmann is very difficult. So far, it has only been solved for  $n \leq 4$ .

The following theorem of Gotzmann is classical [7].

**Theorem 1.2.** *Let  $u \in S_n$ . Then there exists an integer  $\tau_n(u) \geq 0$  such that  $ux_n^s$  is a Gotzmann monomial in  $R_n$  if and only if  $s \geq \tau_n(u)$ .*

For instance,  $u$  is a Gotzmann monomial in  $R_n$  if and only if  $\tau_n(u) = 0$ .

**Definition 1.3.** *We shall refer to the above integer  $\tau_n(u)$  as the Gotzmann threshold of  $u$  in  $R_n$ .*

Determining  $\tau_n(u)$  for all  $u \in S_n$  reduces to determine  $\tau_n(u_0)$  for all  $u_0 \in S_{n-1}$ . Indeed, decomposing  $u = u_0x_n^t$  with  $u_0 \in S_{n-1}$  and  $t \geq 0$ , Theorem 1.2 implies

$$\tau_n(u) = \max(\tau_n(u_0) - t, 0).$$

That is, if  $t \geq \tau_n(u_0)$ , then  $u_0x_n^t$  is a Gotzmann monomial in  $R_n$  whence  $\tau_n(u) = 0$ , whereas if  $t < \tau_n(u_0)$ , then  $ux_n^s = u_0x_n^{t+s}$  is a Gotzmann monomial in  $R_n$  if and only if  $t+s \geq \tau_n(u_0)$  if and only if  $s \geq \tau_n(u_0) - t$ .

Currently, the function  $\tau_n$  is only known for  $n = 2, 3, 4$ . Indeed, we have  $\tau_2(u) = 0$  for all  $u \in S_2$  since every monomial in  $S_2$  is a Gotzmann monomial, as is easily seen. For  $n = 3$ , it follows from results in [15] that for  $u_0 = x_1^a x_2^b \in S_2$ , we have

$$\tau_3(x_1^a x_2^b) = \binom{b}{2}.$$

Finally, the case  $n = 4$  was settled in [5], where for  $u_0 = x_1^a x_2^b x_3^c \in S_3$ , the precise formula for  $\tau_4(u_0)$  turned out to be rather intricate:

$$(1) \quad \tau_4(x_1^a x_2^b x_3^c) = \binom{\binom{b}{2}}{2} + \frac{b+4}{3} \binom{b}{2} + (b+1) \binom{c+1}{2} + \binom{c+1}{3} - c.$$

It is a general fact that the exponent  $a$  of  $x_1$  plays no role for the Gotzmann property [5]. The case where  $a = c = 0$  is the most interesting, being responsible for the main terms of the above formula:

$$\tau_4(x_2^b) = \binom{\binom{b}{2}}{2} + \frac{b+4}{3} \binom{b}{2}.$$

It is an open problem to determine  $\tau_n(u_0)$  for  $n \geq 5$  and all  $u_0 \in S_{n-1}$ . In this paper, we develop a general method to compute  $\tau_n(u_0)$  for any given  $u_0 \in S_{n-1}$ . Based on the above observations for  $n = 4$ , it makes sense to focus on the case  $u_0 = x_2^d$  as a priority. (We switch to exponent  $d$  for the *degree* of the resulting polynomial formula.) As an application of our method, we obtain in Section 5 the formula

$$(2) \quad \tau_5(x_2^d) = \binom{\binom{\binom{d}{2}}{2} + \binom{d+1}{3} + \binom{d}{2}}{2} - \binom{\binom{d}{2}}{3} + \binom{d+3}{4} - d.$$

It remains an open problem to determine  $\tau_n(x_2^d)$  for all  $n \geq 6$ . Based on the cases  $3 \leq n \leq 5$ , we conjecture that  $\tau_n(x_2^d)$  is a polynomial in  $d$  of degree  $2^{n-2}$  and dominant term equal to that of the  $(n-2)$ -iterated binomial coefficient

$$\binom{\binom{\binom{d}{2}}{2}}{\dots 2}.$$

**1.1. Contents.** In Section 2, we recall or introduce basic notions such as the maxgen monomial, gaps and cogaps, in terms of which Gotzmann monomials can be characterized. In Section 3, we develop tools to effectively exploit these characterizations. Our main result to compute the Gotzmann threshold  $\tau_n(u)$  of any monomial  $u \in S_n$  is proved in Section 4. Finally, in Section 5, we apply our method to compute the Gotzmann threshold in the significant case  $u = x_2^d$  in  $S_n$  for  $n = 5$ . We conclude with a conjecture pointing to the expected intricacy of the formula for  $\tau_n(x_2^d)$  for  $n \geq 6$ .

## 2. BACKGROUND

In this section, we recall or introduce concepts with which Gotzmann monomials in  $R_n$  can be characterized.

### 2.1. The maxgen monomial.

**Notation 2.1.** Let  $u \in S_n \setminus \{1\}$ . Let  $k \leq n$  be the largest index such that  $x_k$  divides  $u$ . We then set  $\max(u) = k$  and  $\lambda(u) = x_k$ .

**Definition 2.2.** Let  $B \subseteq S_{n,d}$  be a set of monomials of degree  $d \geq 1$ . The maxgen monomial of  $B$  is defined as

$$\maxgen(B) = \prod_{u \in B} \lambda(u).$$

For the empty set, we define  $\maxgen(\emptyset) = 1$ .

For instance,  $S_{3,2} = \{x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2\}$ , hence  $\maxgen(S_{3,2}) = x_1x_2^2x_3^3$ . Note that  $\deg(\maxgen(B)) = |B|$  for all  $B \subseteq S_{n,d}$ .

### 2.2. Gaps and cogaps.

**Notation 2.3.** Let  $u \in S_{n,d}$ . We denote by  $B(u) \subseteq S_{n,d}$  the smallest set of monomials containing  $u$  and stable under the operations  $v \mapsto vx_i/x_j$  whenever  $x_j$  divides  $v$  and  $i \leq j$ .

Thus,  $B(u)$  is the minimal set of monomial generators of the principal-Borel ideal  $\langle u \rangle$  spanned by  $u$ .

We order the set  $S_{n,d}$  lexicographically, with  $\max(S_{n,d}) = x_1^d$  and  $\min(S_{n,d}) = x_n^d$ . For  $u, v \in S_n$ , we shall write  $u \leq v$  *exclusively* when  $\deg(u) = \deg(v)$  and  $u$  is lexicographically smaller than or equal to  $v$ .

**Notation 2.4.** Let  $u \in S_{n,d} \setminus \{x_1^d\}$ . We denote by  $\text{pred}_n(u)$ , or simply  $\text{pred}(u)$  when the context is clear, the predecessor of  $u$  in  $S_{n,d}$ , i.e. the smallest  $v \in S_{n,d}$  such that  $u < v$ . For  $\ell \geq 1$ , we recursively denote by  $\text{pred}^\ell(u)$  the  $\ell$ th predecessor of  $u$  in  $S_{n,d}$ , provided  $\text{pred}^{\ell-1}(u) \neq x_1^d$ .

Given  $u \in S_{n,d}$ , we will consider upward paths from  $u$  to some larger  $v \in S_{n,d}$ . In that respect, an *elementary step* is the shortest possible upward path from  $u$ , namely from  $u$  to its predecessor  $\text{pred}_n(u)$  in  $S_{n,d}$ .

Recall [5] that if  $m = \max(u) \geq 2$  and  $u = u_0x_m^a$  with  $\max(u_0) \leq m-1$  and  $a \geq 1$ , then

$$(3) \quad \text{pred}_n(u_0x_m^a) = u_0x_{m-1}x_n^{a-1}.$$

**Notation 2.5.** For  $u \leq v \in S_{n,d}$ , we denote

$$\begin{aligned} L_n(u) &= \{z \in S_{n,d} \mid z \geq u\}, \\ L_n^*(v, u) &= \{z \in S_{n,d} \mid v > z \geq u\} = L_n(u) \setminus L_n(v). \end{aligned}$$

The sets  $L_n(u)$  and  $L_n^*(v, u)$  are called *lexsegments* and *lexintervals*, respectively. By a slight abuse of notation, we shall drop the index  $n$  and write  $L(u)$  for  $L_n(u)$  and  $L^*(v, u)$  for  $L_n^*(v, u)$ .

**Notation 2.6.** For  $u \leq v \in S_{n,d}$ , we denote

$$\mu_n(u, v) = \maxgen(L^*(v, u))$$

and call  $\mu_n(u, v)$  the cost of the upward path from  $u$  to  $v$  in  $S_{n,d}$ .

Note that  $\mu_n(u, u) = 1$ . We shall use the *arrow notation*

$$u \xrightarrow{\mu_n(u,v)} v$$

to indicate the upward path from  $u$  to  $v$  in  $S_{n,d}$  together with its cost  $\mu_n(u, v) = \maxgen(L^*(v, u))$ . The most basic step is from  $u$  to  $\text{pred}(u)$ . Thus  $\mu_n(u, \text{pred}(u)) = \lambda(u)$ , the last variable dividing  $u$ . In arrow notation, this is summarized as

$$u \xrightarrow{\lambda(u)} \text{pred}(u).$$

**Example 2.7.** Let  $u = x_2^2 x_4 x_5$ ,  $v = x_2^2 x_3 x_4 \in S_5$ . Then  $v > u$  and the elementary steps of the upward path from  $u$  to  $v$  in  $S_5$  are

$$u = x_2^2 x_4 x_5 \xrightarrow{x_5} x_2^2 x_4^2 \xrightarrow{x_4} x_2^2 x_3 x_5 \xrightarrow{x_5} x_2^2 x_3 x_4 = v.$$

Thus  $\mu_5(u, v) = x_4 x_5^2$ , i.e.  $u \xrightarrow{x_4 x_5^2} v$ .

**Notation 2.8.** Let  $u \in S_{n,d}$ . Noting that  $B(u) \subseteq L(u)$ , we define

$$\begin{aligned} \text{gaps}_n(u) &= L(u) \setminus B(u), \\ \text{cogaps}_n(u) &= L^*(\text{pred}^g(u), u), \text{ where } g = |\text{gaps}_n(u)| \\ &= L^*(\tilde{u}, u), \text{ where } \tilde{u} = \text{pred}^g(u). \end{aligned}$$

**Notation 2.9.** Let  $n \geq 3$ . We denote by  $\text{mg}_n$  and  $\text{mc}_n$  the maps from  $S_n$  to  $S_n$  defined by

$$\begin{aligned} \text{mg}_n(u) &= \maxgen(\text{gaps}_n(u)) \\ \text{mc}_n(u) &= \maxgen(\text{cogaps}_n(u)) \\ &= \mu_n(u, \tilde{u}) \end{aligned}$$

for all  $u \in S_n$ .

**2.3. Characterizations of Gotzmann monomials.** Here is a first characterization of Gotzmann monomials in  $R_n$  in terms of  $\text{mg}_n(u)$  and  $\text{mc}_n(u)$ .

**Theorem 2.10** ([4, 5]). Let  $u \in S_n$ . Then  $u$  is a Gotzmann monomial in  $R_n$  if and only if

$$\text{mg}_n(u) = \text{mc}_n(u).$$

In this paper, we shall mostly use the following more flexible characterization, which does not require one to first determine  $\tilde{u}$  and  $\text{mc}_n(u)$  in order to apply Theorem 2.10.

**Corollary 2.11.** *Let  $u \in S_n$ . Then  $u$  is a Gotzmann monomial in  $R_n$  if and only if there exists  $v \geq u$  such that  $\text{mg}_n(u) = \mu_n(u, v)$ .*

*Proof.* Assume first that  $u \in S_{n,d}$  is a Gotzmann monomial in  $R_n$ . Then  $\text{mg}_n(u) = \text{mc}_n(u) = \mu_n(u, \tilde{u})$ , where  $\tilde{u}$  is introduced in Notation 2.8. Thus  $\text{mg}_n(u) = \mu_n(u, v)$  with  $v = \tilde{u}$ .

Conversely, assume  $\text{mg}_n(u) = \mu_n(u, v)$  for some  $v \geq u$  in  $S_n$ . Let  $g = \deg(\text{mg}_n(u)) = \deg(\mu_n(u, v))$ . Then  $u$  has  $g$  gaps and the upward path from  $u$  to  $v$  has length  $g$ . That is,  $v$  is the  $g$ th predecessor of  $u$  in  $S_n$ , i.e.  $v = \tilde{u}$  by definition of  $\tilde{u}$ . Thus  $\text{mg}_n(u) = \mu_n(u, \tilde{u}) = \text{mc}_n(u)$ , whence  $u$  is a Gotzmann monomial in  $R_n$  by Theorem 2.10.  $\square$

**2.4. The method.** Corollary 2.11 yields the following method to determine  $\tau_n(u_0)$  for  $u_0 \in S_{n-1}$ . Using the tools developed in Section 3, one first computes  $\text{mg}_n(u_0 x_n^t)$ . One then computes costs  $\mu_n(u_0 x_n^t, v)$  of upward paths in  $S_n$  starting from  $u_0 x_n^t$ , and stop if and when such a cost is found to coincide with  $\text{mg}_n(u_0 x_n^t)$ . This justifies the following definition.

**Definition 2.12.** *Given  $u_0 \in S_{n-1}$  and  $t \geq 0$ , we will refer to the monomial  $\text{mg}_n(u_0 x_n^t) \in S_n$  as the target.*

Now, while looking to realize the target  $\text{mg}_n(u_0 x_n^t)$  as the cost of upward paths in  $S_n$  starting from  $u_0 x_n^t$ , we will look to first realize truncated versions of it by ignoring the last variables.

**Notation 2.13.** *Let  $n \geq i \geq 1$  be positive integers. We denote by*

$$\pi_i: S_n \rightarrow S_i$$

*the truncation morphism defined by  $\pi_i(x_1^{a_1} \cdots x_n^{a_n}) = x_1^{a_1} \cdots x_i^{a_i}$ .*

By a slight abuse of notation, we may and will safely ignore the index  $n$  in the notation  $\pi_i$ .

**Definition 2.14.** *Given  $u_0 \in S_{n-1}$ ,  $t \geq 0$  and  $i \leq n$ , we will refer to the monomial  $\pi_i(\text{mg}_n(u_0 x_n^t)) \in S_i$  as the  $i$ -truncated target, or more shortly as the  $i$ -target.*

The effectiveness of the method is illustrated in Section 5 with the determination of formula (2) for the Gotzmann threshold  $\tau_5(x_2^d)$ .

### 3. COMPUTING $\text{mg}_n(u)$ AND $\mu_n(u, v)$

In this Section, we develop tools enabling us to compute  $\text{mg}_n(u)$  for  $u \in S_n$  and costs  $\mu_n(u, v)$  of upward paths in  $S_n$  starting from  $u$ .

3.1. **The map  $\sigma_n$ .** The following map is used everywhere in the sequel.

**Notation 3.1.** We denote by  $\sigma_n: S_n \rightarrow S_n$  the map defined, for all

$$u = \prod_{i=1}^n x_i^{a_i} \in S_n, \text{ by}$$

$$\sigma_n\left(\prod_{i=1}^n x_i^{a_i}\right) = \prod_{i=1}^n x_i^{a_1+\dots+a_i} = x_1^{a_1} x_2^{a_1+a_2} \dots x_n^{a_1+a_2+\dots+a_n}.$$

For instance,  $\sigma_2(x_2) = x_2$ ,  $\sigma_5(x_2) = x_2 x_3 x_4 x_5$  and  $\sigma_4(x_2^2 x_3^5) = x_2^2 x_3^7 x_4^7$ .

A useful property of  $\sigma_n$  is its compatibility with the product.

**Lemma 3.2.** Let  $u, v \in S_n$ . Then  $\sigma_n(uv) = \sigma_n(u)\sigma_n(v)$ .

*Proof.* Set  $u = \prod_{i=1}^n x_i^{a_i}$ ,  $v = \prod_{i=1}^n x_i^{b_i}$ . Then

$$\begin{aligned} \sigma_n(uv) &= \sigma_n\left(\prod_{i=1}^n x_i^{a_i+b_i}\right) = \prod_{i=1}^n x_i^{(a_1+b_1)+\dots+(a_i+b_i)} \\ &= \prod_{i=1}^n x_i^{(a_1+\dots+a_i)+(b_1+\dots+b_i)} = \sigma_n(u)\sigma_n(v). \quad \square \end{aligned}$$

Here is a simple recursive formula.

**Lemma 3.3.** Let  $u_0 \in S_{n-1}$ . For all  $t \geq 0$ , we have

$$\sigma_n(u_0 x_n^t) = \sigma_{n-1}(u_0) \cdot x_n^{t+\deg(u_0)}.$$

*Proof.* Follows from the definition of  $\sigma_n$ , noting that if  $u_0 = x_1^{a_1} \dots x_{n-1}^{a_{n-1}}$ , then

$$\deg(u_0) + t = a_1 + \dots + a_{n-1} + t. \quad \square$$

We will also need to apply  $\sigma_n$  iteratively. Here is the result.

**Proposition 3.4.** Let  $u = \prod_{i=1}^n x_i^{a_i}$ . Then for all  $t \geq 1$ , one has

$$(4) \quad \sigma_n^t(u) = \prod_{i=1}^n x_i^{\sum_{j=1}^i a_j \binom{t-1+i-j}{t-1}}.$$

*Proof.* By induction on  $t$ , using the recurrence relations of the binomial coefficients. For  $t = 1$  we have  $\sum_{j=1}^i a_j \binom{i-j}{0} = a_1 + \dots + a_i$ , as desired. Assume now the formula true for some  $t \geq 1$ . Let us prove it for  $t + 1$ . Set  $b_i = \sum_{j=1}^i a_j \binom{t-1+i-j}{t-1}$ , so that  $\sigma_n^t(u) = \prod_{i=1}^n x_i^{b_i}$  by assumption. Now  $\sigma_n^{t+1}(u) = \sigma_n(\sigma_n^t(u)) = \sigma_n\left(\prod_{i=1}^n x_i^{b_i}\right) = \prod_{i=1}^n x_i^{\sum_{j=1}^i b_j}$ . For all  $i$ ,

set  $c_i = \sum_{j=1}^i b_j$ . We will show that  $c_i = \sum_{j=1}^i a_j \binom{t+i-j}{t}$ , thereby concluding the proof. We have

$$\begin{aligned} c_i &= \sum_{j=1}^i \left( \sum_{r=1}^j a_r \binom{t-1+j-r}{t-1} \right) \\ &= \sum_{r=1}^i a_r \sum_{j=r}^i \binom{t-1+j-r}{t-1} \end{aligned}$$

Setting  $k = j - r$ , we obtain

$$c_i = \sum_{r=1}^i a_r \sum_{k=0}^{i-r} \binom{t-1+k}{t-1}.$$

Using the well-known binomial formula

$$\sum_{k=0}^N \binom{d+k}{d} = \binom{d+1+N}{d+1},$$

and setting  $r = j$ , we obtain

$$c_i = \sum_{j=1}^i a_j \binom{t+i-j}{t},$$

as desired. □

For convenience, here is formula (4) in expanded form:

$$(5) \quad \sigma_n^t(x_1^{a_1} \cdots x_n^{a_n}) = x_1^{a_1} x_2^{a_1 t + a_2} x_3^{a_1 \binom{t+1}{2} + a_2 t + a_3} x_4^{a_1 \binom{t+2}{3} + a_2 \binom{t+1}{2} + a_3 t + a_4} \cdots$$

The following result was first stated as Corollary 6.9 in [5].

**Proposition 3.5.** *Let  $u \in S_n$ . Then*

$$\text{mg}_n(ux_n) = \sigma_n(\text{mg}_n(u)).$$

**Corollary 3.6.** *Let  $u \in S_n$ . For all  $t \geq 0$ , we have*

$$\text{mg}_n(ux_n^t) = \sigma_n^t(\text{mg}_n(u)).$$

**3.2. Comparing maxgen monomials in  $S_n$  and  $S_{n-1}$ .** The following result is Theorem 5.19 in [5]. See Notation 2.3 for the set  $B(u)$ .

**Theorem 3.7.** *Let  $u = x_{i_1} \cdots x_{i_d}$  with  $i_1 \leq \cdots \leq i_d \leq n$ . Then*

$$\text{mg}_n(u) = \prod_{k=1}^{d-1} \left( \prod_{j=i_{k+1}+1}^n x_j^{\binom{d-k-2+j-i_{k+1}}{d-k-1}} \right)^{|B(x_{i_1} \cdots x_{i_k})|-1},$$

where the internal product is set to 1 if  $i_{k+1} = n$ .

**Proposition 3.8.** *Let  $u_0 \in S_{n-1}$ . Then*

$$\pi_{n-1}(\text{mg}_n(u_0)) = \text{mg}_{n-1}(u_0).$$

*Proof.* Write  $u = x_{i_1} \cdots x_{i_d}$  with  $i_1 \leq \cdots \leq i_d \leq n-1$ . By Theorem 3.7 above, we have

$$\text{mg}_n(u) = \prod_{k=1}^{d-1} \left( \prod_{j=i_{k+1}+1}^n x_j^{\binom{d-k-2+j-i_{k+1}}{d-k-1}} \right)^{|B(x_{i_1} \cdots x_{i_k})|-1}.$$

Let us now isolate the variable  $x_n$ . This yields

$$\text{mg}_n(u) = \prod_{k=1}^{d-1} \left( \prod_{j=i_{k+1}+1}^{n-1} x_j^{\binom{d-k-2+j-i_{k+1}}{d-k-1}} \cdot x_n^{\binom{d-k-2+n-i_{k+1}}{d-k-1}} \right)^{|B(x_{i_1} \cdots x_{i_k})|-1}.$$

Again by Theorem 3.7, the left part computes  $\text{mg}_{n-1}(u_0)$ , i.e.

$$\text{mg}_{n-1}(u_0) = \prod_{k=1}^{d-1} \left( \prod_{j=i_{k+1}+1}^{n-1} x_j^{\binom{d-k-2+j-i_{k+1}}{d-k-1}} \right)^{|B(x_{i_1} \cdots x_{i_k})|-1}.$$

Summarizing, we have

$$(6) \quad \text{mg}_n(u) = \text{mg}_{n-1}(u_0) \cdot x_n^{\sum_{k=1}^{d-1} \binom{d-k-2+n-i_{k+1}}{d-k-1} \cdot (|B(x_{i_1} \cdots x_{i_k})|-1)}.$$

This concludes the proof of the proposition.  $\square$

**Corollary 3.9.** *Let  $u_0 \in S_{n-1}$ . For all  $t \in \mathbb{N}$ , we have*

$$\pi_{n-1}(\text{mg}_n(u_0 x_n^t)) = \sigma_{n-1}^t(\text{mg}_{n-1}(u_0)) = \text{mg}_{n-1}(u_0 x_n^t).$$

*Proof.* Follows from Proposition 3.8 and induction on  $t$  using the formula  $\text{mg}_n(ux_n) = \sigma_n(\text{mg}_n(u))$  for any  $u \in S_n$ .  $\square$

**3.3. The polynomial  $f(t)$ .** In this section, to a given monomial  $u_0 \in S_{n-1}$  with  $n \geq 3$ , we attach a polynomial  $f(t)$  which will be involved in our main result towards determining  $\tau_n(u_0)$ .

**Notation 3.10.** *Let  $u_0 \in S_{n-1}$ . For  $t \in \mathbb{N}$ , we denote by  $f(t) \in \mathbb{N}$  the degree in  $x_n$  of the monomial  $\text{mg}_n(u_0 x_n^t) \in S_n$ , i.e.*

$$f(t) = \deg_{x_n}(\text{mg}_n(u_0 x_n^t)).$$

Let us denote

$$(7) \quad w(t) = \pi_{n-1}(\text{mg}_n(u_0 x_n^t)) \in S_{n-1}.$$

Recall from Definitions 2.12 and 2.14 that  $\text{mg}_n(u_0 x_n^t)$  and its truncation  $w(t) \in S_{n-1}$  are called the *target* and the  $(n-1)$ -*target*, respectively.

It follows from the above corollary that

$$w(t) = \text{mg}_{n-1}(u_0 x_{n-1}^t),$$

whence the decomposition

$$(8) \quad \text{mg}_n(u_0 x_n^t) = \text{mg}_{n-1}(u_0 x_{n-1}^t) x_n^{f(t)} = w(t) x_n^{f(t)}.$$

The proof of Proposition 3.8 yields an explicit formula for  $f(t)$ .

**Theorem 3.11.** *Let  $u_0 = x_{i_1} \dots x_{i_r} \in S_{n-1}$  with  $i_1 \leq \dots \leq i_r \leq n-1$  and  $n \geq 3$ . For  $t \geq 0$ , set as above  $\text{mg}_n(u_0 x_n^t) = w(t) x_n^{f(t)}$ . Then*

$$f(t) = \sum_{k=1}^{r-1} \binom{t+r-k-2+n-i_{k+1}}{n-1-i_{k+1}} \cdot (|B(x_{i_1} \dots x_{i_k})| - 1).$$

*Proof.* We have  $\deg(u_0) = r$ . Fixing  $t$ , let  $u = u_0 x_n^t$  and  $d = \deg(u) = r+t$ . Formula (6) and the binomial formula  $\binom{a}{b} = \binom{a}{a-b}$  directly yield the stated result.  $\square$

**Corollary 3.12.** *Fixing  $n \geq 3$  and  $u_0 \in S_{n-1}$  not divisible by  $x_1$ , the above corresponding function  $f(t)$  is a polynomial in  $t$  of degree at most  $n-3$ .*

*Proof.* Writing  $u_0 = x_{i_1} \dots x_{i_r}$  with  $2 \leq i_1 \leq \dots \leq i_r \leq n-1$ , the above formula for  $f(t)$  is clearly a polynomial in  $t$  of degree at most  $n-1-i_{k+1}$  for any  $1 \leq k \leq r-1$ , and hence of degree at most  $n-3$  since  $i_{k+1} \geq 2$  for all  $k$  by hypothesis.  $\square$

**Example 3.13.** *The following formulas were shown in [5]. For  $n=3$ , we have  $\text{mg}_3(x_2^b x_3^t) = x_3^{\binom{b}{2}}$  independently of  $t$ , so that  $f(t) = \binom{b}{2}$  for all  $t$  here. For  $n=4$ , we have  $\text{mg}_4(x_2^b x_3^c x_4^t) = x_3^{\binom{b}{2}} x_4^{f(t)}$  where*

$$f(t) = \binom{b}{2} t + \left(\frac{b+1}{3} + c\right) \binom{b}{2} + (b+1) \binom{c+1}{2} + \binom{c+1}{3} - c,$$

*a polynomial of degree 1. Incidentally, note that*

$$\text{mg}_4(x_2^b x_3^c x_4^t) = \text{mg}_3(x_2^b x_3^c) x_4^{f(t)}.$$

**3.4. Multiplying by  $x_n$ .** Given  $u_0 \in S_{n-1}$ , one needs to repeatedly multiply  $u_0$  by  $x_n$  until obtaining an equality  $\text{mg}_n(u_0 x_n^t) = \mu_n(u_0 x_n^t, v)$  for some  $v \geq u_0 x_n^t$  characterizing the Gotzmann property by Corollary 2.11. In that respect, one needs to compare the costs  $\mu_n(u, v)$  and  $\mu_n(u x_n, v x_n)$ .

**Proposition 3.14.** *Let  $u < v \in S_{n,d}$ . Then  $\mu_n(u x_n, v x_n) = \sigma_n(\mu_n(u, v))$ .*

*Proof.* Let  $\ell$  be the length of the path from  $u$  to  $v$ , that is,

$$\ell = |L^*(v, u)| = \deg(\mu_n(u, v)).$$

Then  $v = \text{pred}^\ell(u)$ . We proceed by induction on  $\ell$ .

**Case  $\ell = 1$ .** Then  $\mu_n(u, v) = x_i$  for some  $2 \leq i \leq n$ , i.e.

$$u \xrightarrow{x_i} v$$

in arrow notation. Hence  $\max(u) = x_i$ , and so  $u = u'x_i^a$  with  $a \geq 1$  and  $\max(u') < \max(u)$ . Recall that

$$\sigma_n(x_i) = x_i x_{i+1} \cdots x_n$$

by definition of  $\sigma_n$ . As  $v = \text{pred}(u)$ , it follows from (3) that

$$v = u'x_{i-1}x_n^{a-1}.$$

In arrow notation:

$$u'x_i^a \xrightarrow{x_i} u'x_{i-1}x_n^{a-1}.$$

We now compute  $\mu_n(ux_n, vx_n)$ , separately for  $i = n$  and  $i < n$ .

If  $i = n$ , then  $ux_n = u'x_n^{a+1}$  and  $vx_n = u'x_{n-1}x_n^a$ . Therefore  $vx_n = \text{pred}(ux_n)$ . In arrow notation, we have

$$ux_n = u'x_n^{a+1} \xrightarrow{x_n} u'x_{n-1}x_n^a = vx_n$$

Thus  $\mu_n(ux_n, vx_n) = \mu_n(u, v) = x_n = \sigma_n(x_n)$ , as desired.

If  $i < n$ , then  $ux_n = u'x_i^a x_n$  and  $vx_n = u'x_{i-1}x_n^a$ . The path from the former to the latter is as follows:

$$ux_n = u'x_i^a x_n \xrightarrow{x_n x_{n-1} \cdots x_{i+1}} u'x_i^a x_i \xrightarrow{x_i} u'x_{i-1}x_n^a = vx_n$$

Thus, by arrow composition, we have  $\mu_n(ux_n, vx_n) = x_i \cdots x_n = \sigma_n(x_i)$ , as desired. This concludes the case  $\ell = 1$ .

**Case  $\ell \geq 2$ .** We assume by induction hypothesis that the statement is true for  $\ell - 1$ . Let  $u' = \text{pred}(u)$ . Then  $v > u' > u$  and  $\mu_n(u, v) = \mu_n(u, u')\mu_n(u', v)$ . Since  $\deg(\mu_n(u, u')) = 1$ , then  $\deg(\mu_n(u', v)) = \ell - 1$ . It follows from the case  $\ell = 1$  and the induction hypothesis that

$$\begin{aligned} \mu_n(ux_n, u'x_n) &= \sigma_n(\mu_n(u, u')), \\ \mu_n(u'x_n, vx_n) &= \sigma_n(\mu_n(u', v)). \end{aligned}$$

By arrow composition and the compatibility of  $\sigma_n$  with the product, we have

$$\begin{aligned}\mu_n(ux_n, vx_n) &= \mu_n(ux_n, u'x_n)\mu_n(u'x_n, vx_n) \\ &= \sigma_n(\mu_n(u, u'))\sigma_n(\mu_n(u', v)) \\ &= \sigma_n(\mu_n(u, u')\mu_n(u', v)) \\ &= \sigma_n(\mu_n(u, v)),\end{aligned}$$

as desired. This completes the proof of the proposition.  $\square$

#### 4. MAIN RESULT

Given  $u_0 \in S_{n-1}$  with  $n \geq 3$ , we attached to it a polynomial  $f(t)$  in Section 3.3. In this section, we further attach to  $u_0$  two more polynomials  $h(t), k(t)$ . With them, our main result is the following expression in Theorem 4.15 for the Gotzmann threshold of  $u_0$  in  $R_n$ :

$$\tau_n(u_0) = f(0) - h(0) - k(0).$$

Several preliminaries are needed for the construction of  $h(t), k(t)$  in Section 4.4. Indeed, in view of reaching the equality  $\text{mg}_n(u_0x_n^t) = \mu_n(u_0x_n^t, v)$  for some  $v \geq u_0x_n^t$ , characterizing the Gotzmann property, we need tools allowing to first reach its truncation in  $S_{n-1}$  using  $\pi_{n-1}$ .

##### 4.1. Truncating the cost $\mu_n$ .

**Proposition 4.1.** *Let  $u_0 < v_0$  in  $S_{n-1,d}$ . Then*

$$\pi_{n-1}(\mu_n(u_0, v_0)) = \mu_{n-1}(u_0, v_0).$$

*That is,  $\mu_n(u_0, v_0) = \mu_{n-1}(u_0, v_0)x_n^s$  for some  $s \geq 0$ .*

*Proof.* In  $S_{n-1,d}$ , consider the full upward path from  $u_0$  to  $v_0$ :

$$u_0 < u_1 < u_2 < \cdots < u_k = v_0,$$

where  $u_{i+1} = \text{pred}_{n-1}(u_i)$  for all  $i$ . Given  $i \leq k-1$ , we have  $u_i = w_i x_r^a$  where  $w_i \in S_{r-1}$  and  $a \geq 1$ . So the upward step from  $u_i$  to  $u_{i+1}$  in  $S_{n-1}$  is given by

$$u_i = w_i x_r^a \xrightarrow{x_r} u_{i+1} = w_i x_{r-1} x_n^{a-1}.$$

That is, we have  $\mu_{n-1}(u_i, u_{i+1}) = x_r$ . Now the predecessor of  $u_i$  in  $S_n$  is given by  $\text{pred}_n(w_i x_r^a) = w_i x_{r-1} x_n^{a-1}$ . Thus, the upward path from  $u_i$  to  $u_{i+1}$  in  $S_n$  is given by

$$u_i = w_i x_r^a \xrightarrow{x_r} w_i x_{r-1} x_n^{a-1} \xrightarrow{x_n^{a-1}} u_{i+1} = w_i x_{r-1} x_n^{a-1}.$$

That is, we have  $\mu_n(u_i, u_{i+1}) = x_r x_n^{a-1} = \mu_{n-1}(u_i, u_{i+1}) x_n^{a-1}$ . Since  $\mu_n(u_0, v_0) = \prod_{i=0}^{k-1} \mu_n(u_i, u_{i+1})$ , the proof is complete.  $\square$

#### 4.2. The monomial $z_n(t)$ .

**Proposition 4.2.** *Let  $u_0 \in S_{n-1}$ . Assume  $t \geq \tau_{n-1}(u_0)$ . Let  $w(t) = \pi_{n-1}(\text{mg}_n(u_0x_n^t))$ . Then there exists  $v(t) \in S_{n-1}$  such that*

$$w(t) = \pi_{n-1}(\mu_n(u_0x_n^t, v(t))).$$

*Proof.* By hypothesis on  $t$ , we have that  $u_0x_{n-1}^t$  is a Gotzmann monomial in  $R_{n-1}$ . Hence by Corollary 2.11, there exists  $v(t) \in S_{n-1}$  such that

$$(9) \quad \text{mg}_{n-1}(u_0x_{n-1}^t) = \mu_{n-1}(u_0x_{n-1}^t, v(t)).$$

Now

$$\begin{aligned} w(t) &= \pi_{n-1}(\text{mg}_n(u_0x_n^t)) \\ &= \text{mg}_{n-1}(u_0x_{n-1}^t) \text{ by Corollary 3.9} \\ &= \mu_{n-1}(u_0x_{n-1}^t, v(t)) \text{ by (9)} \\ &= \pi_{n-1}(\mu_n(u_0x_{n-1}^t, v(t))) \text{ by Proposition 4.1} \\ &= \pi_{n-1}(\mu_n(u_0x_n^t, v(t))). \end{aligned}$$

The last line follows from  $u_0x_n^t \xrightarrow{x_n^t} u_0x_{n-1}^t \xrightarrow{\mu_n(u_0x_{n-1}^t, v(t))} v(t)$ .  $\square$

**Definition 4.3.** *Let  $u_0 \in S_{n-1}$  with  $n \geq 3$ . For  $t \geq 0$ , we denote by  $z_n(t)$ , if it exists, the smallest monomial in  $S_n$  such that  $u_0x_n^t \leq z_n(t)$  and*

$$\pi_{n-1}(\text{mg}_n(u_0x_n^t)) = \pi_{n-1}(\mu_n(u_0x_n^t, z_n(t))).$$

With the above notation, it follows from Proposition 4.2 that  $z_n(t)$  exists whenever  $t \geq \tau_{n-1}(u_0)$ .

**Example 4.4.** *Let  $u_0 = x_2^2 \in S_{n-1}$  with  $n = 4$ . On the one hand, for  $t \geq 0$ , we have  $\text{mg}_4(x_2^2x_4^t) = x_3x_4^{t+1}$  as follows from Theorem 3.7. Hence  $\pi_3(\text{mg}_4(x_2^2x_4^t)) = x_3$ . For  $t \geq 1$ , we claim that  $z_4(t) = x_2^3x_4^{t-1}$ . Indeed, the upward path in  $S_4$  from  $x_2^2x_4^t$  starts like this:*

$$x_2^2x_4^t \xrightarrow{x_4^t} x_2^2x_3^t \xrightarrow{x_3} x_2^3x_4^{t-1}.$$

Thus  $\mu_4(x_2^2x_4^t, x_2^3x_4^{t-1}) = x_3x_4^t$ , so

$$\pi_3(\mu_4(x_2^2x_4^t, x_2^3x_4^{t-1})) = x_3 = \pi_3(\text{mg}_4(x_2^2x_4^t))$$

as desired. However, for  $t = 0$ , we claim that  $z_4(0)$  does not exist. Indeed,  $\pi_3(\text{mg}_4(x_2^2)) = x_3x_4$ , whereas the first step of the upward path in  $S_4$  from  $x_2^2$  is  $x_2^2 \xrightarrow{x_2} x_1x_4$ , whence  $x_2$  divides  $\pi_3(\mu_4(x_2^2, v))$  for all  $v > x_2^2$  in  $S_4$ , implying

$$x_3 = \pi_3(\text{mg}_4(x_2^2)) \neq \pi_3(\mu_4(x_2^2, v)).$$

**Lemma 4.5.** *Let  $u \in S_{n,d}$  such that  $x_n$  divides  $u$ . For all  $v \in S_{n,d}$  such that  $v > u$ , we have that  $\mu_n(u, v)$  is the product of monomials of the form*

$$x_i x_{i+1} \cdots x_n$$

for various indices  $2 \leq i \leq n$ .

*Proof.* Each elementary step in the upward path from  $u$  to  $v$  is of the form

$$v_0 x_j^a \xrightarrow{x_j} v_1 = v_0 x_{j-1} x_n^{a-1}$$

where  $2 \leq j \leq n$ ,  $v_0 \in S_{j-1}$  and  $a \geq 1$ . Therefore, for the next elementary step starting from  $v_1 = v_0 x_{j-1} x_n^{a-1}$ , there are only two possibilities, namely

$$\begin{cases} v_0 x_{j-1} x_n^{a-1} \xrightarrow{x_{j-1}} v_2 = v_0 x_{j-2} & \text{if } a = 1, \\ v_0 x_{j-1} x_n^{a-1} \xrightarrow{x_n} v_2 = v_0 x_{j-1} x_{n-1} x_n^{a-2} & \text{if } a \geq 2. \end{cases}$$

This shows that if the cost of the current elementary step is  $x_j$ , then the cost of the next elementary step is either  $x_{j-1}$  or  $x_n$ .  $\square$

**Proposition 4.6.** *Let  $u, v \in S_n$  such that  $x_n$  divides  $u$  and  $v > u$ . Assume that  $x_k$  divides  $\mu_n(u, v)$  for some  $k \leq n-1$ . Then there exists  $v' \in S_n$  such that  $u \leq v' < v$  and  $\pi_k(\mu_n(u, v)/x_k) = \pi_k(\mu_n(u, v'))$ .*

*Proof.* By hypothesis, there is an upward path of the form

$$u \longrightarrow u_1 \xrightarrow{x_k} u_2 \longrightarrow v.$$

Then  $\mu_n(u, v) = \mu_n(u, u_1) x_k \mu_n(u_2, v)$ . We may assume that  $u_2$  is maximal with the property that  $x_k$  divides  $\mu_n(u, u_2)$ . It follows from Lemma 4.5 that  $\mu_n(u_2, v)$  is only divisible by variables  $x_j$  with  $j > k$ . Hence  $\pi_k(\mu_n(u, v)/x_k) = \pi_k(\mu_n(u, u_1))$ . So  $v' = u_1$  has the required property.  $\square$

**Theorem 4.7.** *Let  $u_0 \in S_{n-1}$  with  $n \geq 3$ . Let  $t \geq \tau_{n-1}(u_0)$ . As in Definition 4.3, let  $z_n(t) \geq u_0 x_n^t$  be the smallest monomial such that*

$$(10) \quad \pi_{n-1}(\text{mg}_n(u_0 x_n^t)) = \pi_{n-1}(\mu_n(u_0 x_n^t, z_n(t))).$$

Then  $z_n(t+1) = z_n(t) x_n$ .

*Proof.* Denote  $w(t) = \pi_{n-1}(\text{mg}_n(u_0 x_n^t))$ , the current  $(n-1)$ -target.

**Case 1:**  $w(t) = 1$ . That is,  $\text{mg}_n(u_0 x_n^t)$  is a power of  $x_n$ . Then the monomial  $z_n(t) = u_0 x_n^t$  satisfies the required properties, since

$$w(t) = 1 = \mu_n(u_0 x_n^t, u_0 x_n^t) = \pi_{n-1}(\mu_n(u_0 x_n^t, u_0 x_n^t)).$$

In this case, we clearly have  $z_n(t+1) = z_n(t) x_n$ .

**Case 2:**  $\deg(w(t)) \geq 1$ . By Corollary 3.9, we have

$$w(t) = \pi_{n-1}(\text{mg}_n(u_0x_n^t)) = \text{mg}_{n-1}(u_0x_{n-1}^t),$$

whence  $x_{n-1}$  divides  $w(t)$ . Since  $t \geq \tau_{n-1}(u_0)$ , there exists  $v'(t) \in S_{n-1}$  such that

$$w(t) = \mu_{n-1}(u_0x_{n-1}^t, v'(t)).$$

That is, we have the following upward path in  $S_{n-1}$ :

$$u_0x_{n-1}^t \xrightarrow{w(t)} v'(t).$$

For the upward path from  $u_0x_{n-1}^t$  to  $v'(t)$  in  $S_n$ , Proposition 4.1 implies

$$\mu_n(u_0x_{n-1}^t, v'(t)) = w(t)x_n^s$$

for some  $s \in \mathbb{N}$ . Starting from  $u_0x_n^t$ , we get

$$u_0x_n^t \xrightarrow{x_n^t} u_0x_{n-1}^t \xrightarrow{w(t)x_n^s} v'(t)$$

with cumulated cost  $\mu_n(u_0x_n^t, v'(t)) = w(t)x_n^{g(t)}$  where  $g(t) = t + s$ . By Proposition 4.6 and its proof, since  $x_{n-1}$  divides  $w(t)$ , the path

$$u_0x_n^t \xrightarrow{w(t)x_n^{t+s}} v'(t)$$

decomposes as

$$u_0x_n^t \xrightarrow{\frac{w(t)}{x_{n-1}}x_n^{g''(t)}} u''(t) \xrightarrow{x_{n-1}} u'(t) \xrightarrow{x_n^{g'(t)}} v'(t)$$

where  $g'(t) + g''(t) = g(t)$ . It follows that  $\max(u'') = x_{n-1}$  and that, in this path, the elementary step leading to  $u''$  is of the form

$$u'''(t) \xrightarrow{x_n} u''(t),$$

i.e. with  $\text{pred}_n(u'''(t)) = u''(t)$ . In particular,  $u'''(t)$  is divisible by  $x_n$ . Let us denote

$$u'''(t) = v(t)x_n^{a_n}$$

with  $v(t) \in S_{n-1}$  and  $a_n \geq 1$ . Stopping at  $v(t)x_n^{a_n}$ , we obtain the path

$$u_0x_n^t \xrightarrow{\frac{w(t)}{x_{n-1}}x_n^{l(t)}} v(t)x_n^{a_n}$$

with  $l(t) = g''(t) - 1$ . Continuing from here, we have

$$v(t)x_n^{a_n} \xrightarrow{x_n^{a_n}} v(t)x_{n-1}^{a_n}.$$

The cumulated cost so far is the product  $\frac{w(t)}{x_{n-1}}x_n^{l(t)+a_n}$  which is not yet on target. But with just one more step, namely

$$v(t)x_{n-1}^{a_n} \xrightarrow{x_{n-1}} v(t)x_{n-2}x_n^{a_n-1},$$

the cumulated cost reaches  $w(t)x_n^{l(t)+a_n}$ , on target for first time relatively to  $\pi_{n-1}$ . Hence

$$z_n(t) = v(t)x_{n-2}x_n^{a_n-1}.$$

We now multiply the extremities of the initial path by  $x_n$ . By Proposition 3.14, the cost of the new path is given as follows:

$$u_0x_n^{t+1} \xrightarrow{\sigma_n\left(\frac{w(t)}{x_{n-1}}x_n^{l(t)}\right)} v(t)x_n^{a_n+1}.$$

Continuing from here, we have

$$(11) \quad v(t)x_n^{a_n+1} \xrightarrow{x_n^{a_n+1}} v(t)x_{n-1}^{a_n+1},$$

with cumulated cost so far given by

$$(12) \quad \sigma_n\left(\frac{w(t)}{x_{n-1}}x_n^{l(t)}\right)x_n^{a_n+1}.$$

We will now simplify (12) as

$$(13) \quad \sigma_n\left(\frac{w(t)}{x_{n-1}}x_n^{l(t)}\right)x_n^{a_n+1} = \frac{w(t+1)}{x_{n-1}} \cdot x_n^*$$

using the next two claims.

**Claim 1.**  $w(t+1) = \sigma_{n-1}(w(t))$ .

Indeed, we have

$$\begin{aligned} w(t+1) &= \pi_{n-1}(\text{mg}_n(u_0x_n^{t+1})) \\ &= \pi_{n-1}(\sigma_n(\text{mg}_n(u_0x_n^t))) \quad \text{by Prop. 3.5} \\ &= \pi_{n-1}(\sigma_n(w(t)x_n^{f(t)})) \quad \text{by (8)} \\ &= \pi_{n-1}(\sigma_{n-1}(w(t))x_n^{f(t)+\deg w(t)}) \quad \text{by Lemma. 3.3} \\ &= \sigma_{n-1}(w(t)). \end{aligned}$$

**Claim 2.**  $\sigma_{n-1}\left(\frac{w(t)}{x_{n-1}}\right) = \frac{w(t+1)}{x_{n-1}}$ .

Indeed,  $w(t) = \frac{w(t)}{x_{n-1}} \cdot x_{n-1}$ . Applying  $\sigma_{n-1}$  to both sides, using Lemma 3.2, and the formula  $\sigma_{n-1}(x_{n-1}) = x_{n-1}$ , we get

$$\begin{aligned} \sigma_{n-1}(w(t)) &= \sigma_{n-1}\left(\frac{w(t)}{x_{n-1}}\right) \cdot \sigma_{n-1}(x_{n-1}) \\ &= \sigma_{n-1}\left(\frac{w(t)}{x_{n-1}}\right) \cdot x_{n-1}. \end{aligned}$$

Hence

$$(14) \quad \begin{aligned} \sigma_{n-1}\left(\frac{w(t)}{x_{n-1}}\right) &= \frac{\sigma_{n-1}(w(t))}{x_{n-1}} \\ &= \frac{w(t+1)}{x_{n-1}} \text{ by Claim 1.} \end{aligned}$$

This settles Claim 2.

We are now in a position to prove (13):

$$\begin{aligned} \sigma_n\left(\frac{w(t)}{x_{n-1}}x_n^{l(t)}\right)x_n^{a_n+1} &= \frac{\sigma_{n-1}(w(t))}{x_{n-1}}x_n^{\deg(w(t))+l(t)+a_n} \text{ by (14)} \\ &= \frac{w(t+1)}{x_{n-1}} \cdot x_n^* \text{ by Claim 2.} \end{aligned}$$

Note that the cost given by (13) is not yet on target relative to  $\pi_{n-1}$ . Going back to the path with that cost, we had reached  $v(t)x_{n-1}^{a_n+1}$  in (11). Starting from there, the next step upwards is

$$v(t)x_{n-1}^{a_n+1} \xrightarrow{x_{n-1}} v(t)x_{n-2}x_n^{a_n}.$$

Now, the cumulated cost of the extended path is given by

$$\frac{w(t+1)}{x_{n-1}} \cdot x_n^* \cdot x_{n-1} = w(t+1) \cdot x_n^*.$$

This is the first time we reach the desired  $(n-1)$ -target  $w(t+1)$ . Hence  $z_n(t+1)$  is given by the extremity of that extended path, namely  $v(t)x_{n-2}x_n^{a_n+1}$ . Hence

$$\begin{aligned} z_n(t+1) &= v(t)x_{n-2}x_n^{a_n} \\ &= v(t)x_{n-2}x_n^{a_n-1} \cdot x_n \\ &= z_n(t) \cdot x_n, \end{aligned}$$

as desired.  $\square$

**4.3. Partial conversions.** Given  $v \in S_{m-1}$ , we need to know the cost of the path from  $vx_m^k$  to  $vx_{m-1}^\ell x_m^{k-\ell}$  for  $\ell \leq k$ , where we only partially convert some of the  $x_m$  into  $x_{m-1}$ . Here is the required formula.

**Proposition 4.8.** *Let  $v \in S_{m-1}$  with  $2 \leq m \leq n$ . For any  $1 \leq \ell \leq k$ , we have*

$$\mu_n(vx_m^k, vx_{m-1}^\ell x_m^{k-\ell}) = x_m^\ell x_{m+1}^{\binom{k}{2} - \binom{k-\ell}{2}} x_{m+2}^{\binom{k+1}{3} - \binom{k+1-\ell}{3}} \dots x_n^{\binom{k+n-m-1}{n-m+1} - \binom{k+n-m-1-\ell}{n-m+1}}.$$

*Proof.* By arrow composition, we have

$$vx_m^k \longrightarrow vx_{m-1}^\ell x_m^{k-\ell} \longrightarrow vx_{m-1}^k.$$

The respective costs of the long path and of the rightmost one are known. The cost of the leftmost one is given by their quotient, namely

$$\mu_n(vx_m^k, vx_{m-1}x_m^{k-\ell}) = \mu_n(vx_m^k, vx_{m-1}^k) / \mu_n(vx_{m-1}x_m^{k-\ell}, vx_{m-1}^k).$$

Applying the relevant formula for the numerator and the denominator yields the stated result.  $\square$

**Lemma 4.9.** *Let  $v_2 < v_1 \in S_{n,d}$ . Then  $\mu_n(v_2, v_1) = x_n^b$  for some  $b \in \mathbb{N}$  if and only if  $\deg_{x_n}(v_2) \geq b$ .*

*Proof.* Let  $a = \deg_{x_n}(v_2)$ , so that  $v_2 = v'_2 x_n^a$  with  $v'_2 \in S_{n-1}$ . Then  $\text{pred}^i(v_2) = v'_2 x_{n-1}^i x_n^{a-i}$  for all  $0 \leq i \leq a$ . In particular, for  $i = a$ , the monomial  $\text{pred}^a(v_2)$  has max index equal to  $n - 1$ . Hence  $\mu_n(v_2, \text{pred}^{a+1}(v_2)) = x_{n-1} x_n^a$ . Therefore  $v_1 < \text{pred}^{a+1}(v_2)$ , i.e.  $v_1 = \text{pred}^j(v_2)$  for some  $j \leq a$ .  $\square$

**4.4. The polynomials  $f(t), h(t), k(t)$ .** Let  $u_0 \in S_{n-1}$  with  $n \geq 3$ . Having already attached to  $u_0$  the polynomial  $f(t)$  in Section 3.3, we will now attach to it two more polynomials  $h(t), k(t)$  involved in our formula for  $\tau_n(u_0)$  in Theorem 4.15.

**Notation 4.10.** *Let  $u_0 \in S_{n-1}$ . For  $t \geq 0$ , let us denote*

$$w(t) = \pi_{n-1}(\text{mg}_n(u_0 x_n^t)).$$

*Recall that we refer to  $w(t) \in S_{n-1}$  as the  $(n - 1)$ -target.*

Hence

$$(15) \quad \text{mg}_n(u_0 x_n^t) = w(t) x_n^{f(t)},$$

where  $f(t) = \deg_{x_n} \text{mg}_n(u_0 x_n^t)$  was shown to be a polynomial in  $t$  in Section 3.3.

We have seen in Section 4.2 that for  $t \geq \tau_{n-1}(u_0)$ , there exists a smallest monomial  $z_n(t) \geq u_0 x_n^t$  such that

$$(16) \quad \pi_{n-1}(\mu_n(u_0 x_n^t, z_n(t))) = w(t).$$

**Definition 4.11.** *With  $u_0 \in S_{n-1}$  and the above notation, we define  $h(t), k(t)$  for  $t \geq \tau_{n-1}(u_0)$  as follows:*

$$(17) \quad \begin{aligned} h(t) &= \deg_{x_n} \mu_n(u_0 x_n^t, z_n(t)), \\ k(t) &= \deg_{x_n} z_n(t). \end{aligned}$$

We will show that  $h(t)$  and  $k(t)$  are polynomials in  $t$ . Let us first compare  $f(t)$  and  $h(t)$ .

**Proposition 4.12.** *Let  $u_0 \in S_{n-1}$ . Then for all  $t \geq \tau_{n-1}(u_0)$ , we have*

$$f(t+1) - f(t) = \deg(w(t)) = h(t+1) - h(t).$$

*Proof.* First for  $f(t)$ . On the one hand, we have

$$\text{mg}_n(u_0x_n^{t+1}) = w(t+1)x_n^{f(t+1)}.$$

On the other hand,

$$\begin{aligned} \text{mg}_n(u_0x_n^{t+1}) &= \sigma_n(\text{mg}_n(u_0x_n^t)) \\ &= \sigma_n(w(t)x_n^{f(t)}) \\ &= \sigma_{n-1}(w(t))x_n^{f(t)+\deg(w(t))}. \end{aligned}$$

Since  $\sigma_{n-1}(w(t)) \in S_{n-1}$ , it follows by comparing the exponents of  $x_n$  in these expressions for  $\text{mg}_n(u_0x_n^{t+1})$  that  $f(t+1) = f(t) + \deg(w(t))$ , as claimed.

Now for  $h(t)$ . On the one hand, we have

$$\mu_n(u_0x_n^{t+1}, z_n(t+1)) = w(t+1)x_n^{h(t+1)}.$$

On the other hand, we proved above that  $z_n(t+1) = z_n(t)x_n$ . Hence

$$\begin{aligned} \mu_n(u_0x_n^{t+1}, z_n(t+1)) &= \mu_n(u_0x_n^t x_n, z_n(t)x_n) \\ &= \sigma_n(\mu_n(u_0x_n^t, z_n(t))) \\ &= \sigma_n(w(t)x_n^{h(t)}) \\ &= \sigma_{n-1}(w(t))x_n^{h(t)+\deg(w(t))}. \end{aligned}$$

Again, since  $\sigma_{n-1}(w(t)) \in S_{n-1}$ , and by comparing the exponent of  $x_n$  on these expressions of  $\mu_n(u_0x_n^{t+1}, z_n(t+1))$ , we find that  $h(t+1) = h(t) + \deg(w(t))$ , as claimed.  $\square$

**Corollary 4.13.** *The functions  $h(t)$  and  $k(t)$  associated to  $u_0 \in S_{n-1}$  are polynomials in  $t$  for  $t \geq \tau_{n-1}(u_0)$ .*

*Proof.* Since  $f(t)$  is a polynomial in  $t$  as shown in Section 3.3, the function  $t \mapsto f(t+1) - f(t)$ , i.e. the discrete derivative of  $f(t)$ , is also a polynomial in  $t$ , of degree one less than the degree of  $f(t)$ . Since the map  $t \mapsto h(t+1) - h(t) = f(t+1) - f(t)$  is a polynomial in  $t$ , it follows that its discrete primitive  $h(t)$  is itself a polynomial in  $t$ , of degree one more, i.e. of the same degree as  $f(t)$ . Finally, since  $z_n(t+1) = z_n(t)x_n$  for  $t$  large enough, and since  $k(t) = \deg_{x_n} z(t)$ , we have  $k(t+1) - k(t) = 1$  for  $t$  large enough. Hence  $k(t)$  is a polynomial in  $t$  of degree 1 with dominant coefficient 1.  $\square$

**4.5. Determining  $\tau_n(u_0)$ .** Let  $u_0 \in S_{n-1}$  with  $n \geq 3$ . We now use the above polynomials  $f(t), h(t), k(t)$  attached to  $u_0$  and (15), (16), (17) to determine when  $u_0x_n^t$  is a Gotzmann monomial in  $R_n$ .

**Theorem 4.14.** *Let  $u_0 \in S_{n-1}$  with  $n \geq 3$ . Let  $t \in \mathbb{N}$ . Then  $u_0x_n^t$  is a Gotzmann monomial in  $R_n$  if and only if*

$$k(t) \geq f(t) - h(t).$$

*Proof.* Given  $t \geq 0$ , let  $W(t) = \text{mg}_n(u_0x_n^t)$ . Thus

$$W(t) = w(t)x_n^{f(t)}.$$

As in [5], denote  $\tilde{u}(t) = \widetilde{u_0x_n^t} \in S_n$  the unique monomial such that  $|L(\tilde{u}(t))| = |B(u_0x_n^t)|$ , i.e.

$$(18) \quad \tilde{u}(t) = \text{pred}^{|\text{gaps}(u_0x_n^t)|}(u_0x_n^t).$$

Assume first that  $u_0x_n^t$  is a Gotzmann monomial in  $R_n$ . Then we have

$$W(t) = \text{mg}_n(u_0x_n^t) = \text{mg}_n(L^*(\tilde{u}(t), u_0x_n^t)).$$

By the minimality of  $z_n(t)$ , we have

$$z_n(t) \leq \tilde{u}(t).$$

By arrow composition, the arrow

$$u_0x_n^t \xrightarrow{\mu_n(u_0x_n^t, \tilde{u}(t))} \tilde{u}(t)$$

decomposes as

$$u_0x_n^t \xrightarrow{\mu_n(u_0x_n^t, z_n(t))} z_n(t) \xrightarrow{\mu_n(z_n(t), \tilde{u}(t))} \tilde{u}(t).$$

Hence

$$(19) \quad \mu_n(z_n(t), \tilde{u}(t)) = \frac{\mu_n(u_0x_n^t, \tilde{u}(t))}{\mu_n(u_0x_n^t, z_n(t))}.$$

Now, we have

$$\begin{aligned} \mu_n(u_0x_n^t, \tilde{u}(t)) &= w_0(t)x_n^{f(t)}, \\ \mu_n(u_0x_n^t, z_n(t)) &= w_0(t)x_n^{h(t)}. \end{aligned}$$

Hence, by (19), we get

$$\mu_n(z_n(t), \tilde{u}(t)) = x_n^{f(t)-h(t)}.$$

It follows from Lemma 4.9 that  $\deg_{x_n} z_n(t) = k(t) \geq f(t) - h(t)$ .

Conversely, assume  $k(t) \geq f(t) - h(t)$ . Hence  $z_n(t)$  is divisible by  $x_n^{f(t)-h(t)}$ . By Lemma 4.9 again, taking as many predecessors, we have

$$z_n(t) \xrightarrow{x_n^{f(t)-h(t)}} v(t)$$

for some monomial  $v(t) \in S_n$ . Thus we have

$$u_0 x_n^t \xrightarrow{w(t)x_n^{h(t)}} z_n(t) \xrightarrow{x_n^{f(t)-h(t)}} v(t),$$

which by arrow composition yields

$$u_0 x_n^t \xrightarrow{w(t)x_n^{f(t)}} v(t).$$

Let  $g(t) = \deg(w(t)x_n^{f(t)})$ . Hence  $v(t) = \text{pred}^{g(t)}(u_0 x_n^t)$ . Since  $w(t)x_n^{f(t)} = W(t) = \text{mg}_n(u_0 x_n^t)$ , it follows that

$$g(t) = |\text{gaps}(u_0 x_n^t)|.$$

Hence  $v(t) = \tilde{u}(t)$  by definition of  $\tilde{u}(t)$ . Recall that  $L^*(\tilde{u}(t), u_0 x_n^t) = \text{cogaps}(u_0 x_n^t)$  by definition. Therefore

$$\text{mc}_n(u_0 x_n^t) = \text{maxgen}(L^*(\tilde{u}(t), u_0 x_n^t)) = w(t)x_n^{f(t)}.$$

We conclude that

$$\text{mc}_n(u_0 x_n^t) = \text{mg}_n(u_0 x_n^t).$$

Hence  $u_0 x_n^t$  is a Gotzmann monomial in  $R_n$ , as desired.  $\square$

We are now in a position to determine the Gotzmann threshold of  $u_0$  in  $R_n$ .

**Theorem 4.15.** *Let  $u_0 \in S_{n-1}$  with  $n \geq 3$ . Let  $f(t), h(t), k(t)$  be the polynomials attached to  $u_0$ . Then  $\tau_n(u_0) = f(0) - h(0) - k(0)$ .*

*Proof.* Assume  $t \geq \tau_{n-1}(u_0)$ .

**Step 1.** Then  $z_n(t+1) = z_n(t)x_n$  by Proposition 4.2.

**Step 2.** We have  $k(t) = t + k(0)$  for some constant  $k(0)$ . Indeed,  $k(t) = \deg_{x_n} z_n(t)$  by Definition 4.11. It follows from Step 1 that  $k(t+1) = k(t) + 1$  for  $t \geq \tau_{n-1}(u_0)$ . Hence, viewing  $k(t)$  as a polynomial in  $t$ , it is of degree 1 with dominant coefficient 1. Thus  $k(t) = t + k(0)$  for some constant  $k(0)$ .

**Step 3.** By Theorem 4.14, we have that  $u_0 x_n^t$  is a Gotzmann monomial in  $R_n$  if and only if  $k(t) \geq f(t) - h(t)$ .

**Step 4.** By Proposition 4.12, we have  $f(t+1) - f(t) = h(t+1) - h(t) = \deg(w(t))$ .

**Step 5.** By Step 4, it follows that both  $f(t)$  and  $h(t)$  are polynomials in  $t$ , since  $\deg(w(t))$  is a polynomial in  $t$  by definition of  $w(t)$  and

Corollary 3.9. Hence the expressions  $f(0), h(0)$  are well defined, and  $f(t) - h(t) = f(0) - h(0)$  by Step 4 again.

**Step 6.** By Step 3,  $u_0 x_n^t$  is a Gotzmann monomial in  $R_n$  if and only if  $k(t) \geq f(t) - h(t)$ . By Step 5, this holds if and only if  $k(t) \geq f(0) - h(0)$ , and by Step 2, this holds if and only if  $t + k(0) \geq f(0) - h(0)$ . Hence  $\tau_n(u_0) = f(0) - h(0) - k(0)$  by Theorem 1.2. This concludes the proof of the theorem.  $\square$

**4.6. An example.** Let  $n = 5$  and  $u_0 = x_2^2 x_4 \in S_4$ . As a first illustration of the method, before its application to the more demanding computation of  $\tau_5(x_2^d)$  in Section 5, let us show that  $\tau_5(u_0) = 6$ . We do so by determining the attached polynomials  $f(t), h(t), k(t)$  and then using the formula  $\tau_5(u_0) = f(0) - h(0) - k(0)$ .

- Applying Theorem 3.7 to  $u_0 = x_2^2 x_4 = x_{i_1} x_{i_2} \cdots x_{i_d}$  with  $d = 3$  and  $i_1 = i_2 = 2, i_3 = 4$ , we get

$$\text{mg}_5(x_2^2 x_4) = x_3 x_4^2 x_5^5.$$

- Corollary 3.6 then implies

$$\text{mg}_5(x_2^2 x_4 x_5^t) = x_3 x_4^{t+2} x_5^{\binom{t+1}{2} + 2t + 5}.$$

Hence  $f(t) = \binom{t+1}{2} + 2t + 5$ .

- We have  $w(t) = \pi_4(\text{mg}_5(x_2^2 x_4 x_5^t)) = x_3 x_4^{t+2}$ . So, we now seek the smallest monomial  $z_5(t) \geq x_2^2 x_4 x_5^t$  such that

$$(20) \quad \pi_4(\mu_5(u_0 x_5^t, z_5(t))) = x_3 x_4^{t+2}.$$

To do that, we begin the upward path starting from  $u_0 x_5^t = x_2^2 x_4 x_5^t$  and stop just before attaining the 4-target  $x_3 x_4^{t+2}$  given by (20):

$$\begin{aligned} x_2^2 x_4 x_5^t &\xrightarrow{x_5^t} x_2^2 x_4^{t+1} \xrightarrow{x_4} x_2^2 x_3 x_5^t \xrightarrow{x_5^t} x_2^2 x_3 x_4^t \xrightarrow{x_4} x_2^2 x_3^2 x_5^{t-1} \xrightarrow{x_5^{t-1}} \\ x_2^2 x_3^2 x_4^{t-1} &\xrightarrow{x_4} x_2^2 x_3^3 x_5^{t-2} \xrightarrow{x_5^{t-2}} \dots \xrightarrow{x_5} x_2^2 x_3^t x_4 \xrightarrow{x_4} x_2^2 x_3^{t+1} \xrightarrow{x_3} \\ x_2^3 x_5^t &\xrightarrow{x_5^t} x_2^3 x_4^t. \end{aligned}$$

At this point, the cost of the upward path from  $x_2^2 x_4 x_5^t$  to  $x_2^3 x_4^t$ , i.e. the product of the costs of each arrow, is given by

$$\mu_5(x_2^2 x_4 x_5^t, x_2^3 x_4^t) = x_3 x_4^{t+1} x_5^{t+t+(t-1)+\dots+1}.$$

That is, mapping  $x_5$  to 1, we obtain

$$\pi_4(\mu_5(x_2^2 x_4 x_5^t, x_2^3 x_4^t)) = x_3 x_4^{t+1},$$

just below the 4-target (20). Reaching that target for the first time is achieved with one more elementary step:

$$x_2^3 x_4^t \xrightarrow{x_4} x_2^3 x_3 x_5^{t-1}.$$

Hence  $x_2^3 x_3 x_5^{t-1}$  is the smallest monomial such that  $x_2^3 x_3 x_5^{t-1} \geq x_2^2 x_4 x_5^t$  and

$$\pi_4(\mu_5(u_0 x_5^t, x_2^3 x_3 x_5^{t-1})) = x_3 x_4^{t+2}$$

as required by (20). We conclude that

$$z_5(t) = x_2^3 x_3 x_5^{t-1}.$$

- We are now in a position to determine the polynomials  $h(t), k(t)$ :

$$h(t) = \deg_{\mathfrak{S}_{x_5}} \mu_5(x_2^2 x_4 x_5^t, z_5(t)) = \binom{t+3}{2} - 3,$$

and  $k(t) = \deg_{\mathfrak{S}_{x_5}} z(t) = t - 1$ . Hence  $\tau_5(x_2^2 x_4) = f(0) - h(0) - k(0) = 6$ .

### 5. APPLICATION: THE CASE $u_0 = x_2^d$

We now apply the above theory to determine the Gotzmann threshold of the monomial  $u_0 = x_2^d$  in  $S_5$  as a function of  $d$ . While our method works for any monomial in  $S_{n-1}$ , this particular case already concentrates all the complexity of the task while avoiding formulas with too many parameters as the one in (1) for  $\tau_4(x_1^a x_2^b x_3^c)$ .

**Proposition 5.1.** *For all  $n \geq 3$ , we have*

$$\text{mg}_n(x_2^d) = x_3^{\binom{d}{2}} x_4^{\binom{d+1}{3}} x_5^{\binom{d+2}{4}} \cdots x_n^{\binom{d+n-3}{n-1}}.$$

*Proof.* Follows from Theorem 3.7 and induction on  $d$ , using the following summation formula [12]:

$$\sum_{k=1}^{d-1} k \binom{d+n-k-4}{n-3} = \binom{d+n-3}{n-1}.$$

□

Recall that the Gotzmann thresholds of  $x_2^d$  in  $S_3$  and  $S_4$  are given by the formulas

$$\begin{aligned} \tau_3(x_2^d) &= \binom{d}{2}, \\ \tau_4(x_2^d) &= \binom{\binom{d}{2}}{2} + \frac{d+4}{3} \binom{d}{2}, \end{aligned}$$

from [15] and [5], respectively. Here is the main result of this section.

**Theorem 5.2.** *For all  $d \geq 2$ , the Gotzmann threshold of  $x_2^d$  in  $R_5$  is given by*

$$\tau_5(x_2^d) = \left( \binom{\binom{d}{2}}{2} + \binom{d+1}{3} + \binom{d}{2} \right) - \binom{\binom{d}{2}}{3} + \binom{d+3}{4} - d.$$

*Proof.* Fix  $d \geq 2$ . By the previous proposition for  $n = 5$ , we have

$$\text{mg}_5(x_2^d) = x_3^{f_3} x_4^{f_4} x_5^{f_5},$$

where

$$\begin{aligned} f_3 &= \binom{d}{2}, \\ f_4 &= \binom{d+1}{3}, \\ f_5 &= \binom{d+2}{4}. \end{aligned}$$

Now, given  $t \geq 0$ , we have

$$\text{mg}_5(x_2^d x_5^t) = \sigma_5^t(\text{mg}_5(x_2^d)) = x_3^{f_3(t)} x_4^{f_4(t)} x_5^{f_5(t)},$$

where

$$\begin{aligned} f_3(t) &= f_3 = \binom{d}{2}, \\ f_4(t) &= f_3 t + f_4, \\ f_5(t) &= f_3 \binom{t+1}{2} + f_4 t + f_5. \end{aligned}$$

As a first step, we seek the smallest  $t$  so that a monomial  $z_4(t)$  exists with the property  $z_4(t) > x_2^d x_5^t$  and  $\pi_3(\mu_5(x_2^d x_5^t, z_4(t))) = x_3^{f_3}$ . The various steps of the required path are as follows:

$$\begin{array}{ccc} x_2^d x_5^t & \xrightarrow{x_5^t} & x_2^d x_4^t \\ x_2^d x_4^t & \xrightarrow{x_4^t x_5^{\binom{t}{2}}} & x_2^d x_3^t \\ x_2^d x_3^t & \xrightarrow{x_3^{f_3-1} x_4^{\binom{t}{2} - (t-f_3+1)} x_5^{\binom{t+1}{3} - (t-f_3+2)}} & x_2^{d+f_3-1} x_3^{t-f_3+1}. \end{array}$$

The cost of the last step is justified by Proposition 4.8. At this point, the 3-target  $x_3^{f_3}$  is almost reached. One last step will allow to reach that target:

$$x_2^{d+f_3-1} x_3^{t-f_3+1} \xrightarrow{x_3} x_2^{d+f_3} x_5^{t-f_3}.$$

Hence  $z_4(t) = x_2^{d+f_3} x_5^{t-f_3}$ , since this is the very first time we reach the 3-target  $x_3^{f_3}$ . The cost of the upward path from  $x_2 x_5^t$  to  $z_4(t)$  is given by the product of the respective costs of each of the above steps, i.e.

$$\begin{aligned} \mu_5(x_2^d x_5^t, z_4(t)) &= x_3^{f_3} x_4^{t+\binom{t}{2}-\binom{t-f_3+1}{2}} x_5^{t+\binom{t}{2}+\binom{t+1}{3}-\binom{t-f_3+2}{3}} \\ &= x_3^{f_3} x_4^{\binom{t+1}{2}-\binom{t-f_3+1}{2}} x_5^{\binom{t+2}{3}-\binom{t-f_3+2}{3}}. \end{aligned}$$

Set  $h_4(t) = \binom{t+1}{2} - \binom{t-f_3+1}{2}$ . Thus

$$(21) \quad \mu_5(x_2^d x_5^t, z_4(t)) = x_3^{f_3} x_4^{h_4(t)} x_5^{\binom{t+2}{3}-\binom{t-f_3+2}{3}}.$$

By minimality of  $z_4(t)$ , we have  $h_4(t) \leq f_4(t)$ . We know that the difference  $f_4(t) - h_4(t)$  is constant. Let us call this constant  $\delta_4$ , so that

$$\delta_4 = f_4(t) - h_4(t) = f_4(0) - h_4(0).$$

Since  $h_4(0) = \binom{1-f_3}{2}$ , we have

$$\delta_4 = \binom{d+1}{3} + \binom{1-f_3}{2}.$$

Using the formula  $\binom{-a}{2} = \binom{a+1}{2}$ , we get

$$\delta_4 = \binom{d+1}{3} + \binom{f_3}{2}.$$

Using the formulas  $\tau_3(x_2^d) = f_3 = \binom{d}{2}$  and  $\delta_4 = \tau_4(x_2^d) - \tau_3(x_2^d)$ , we recover the formula from [5], namely

$$\tau_4(x_2^d) = \binom{d}{2} + \binom{d+1}{3} + \binom{\binom{d}{2}}{2}.$$

Using  $h_4(t) = f_4(t) - \delta_4$ , formula (21) becomes

$$\mu_5(x_2^d x_5^t, z_4(t)) = x_3^{f_3} x_4^{f_4(t)-\delta_4} x_5^{\binom{t+2}{3}-\binom{t-f_3+2}{3}}.$$

Our next task is to find the smallest monomial  $z_5(t)$  with the property

$$\pi_4(\mu_5(x_2^d x_5^t, z_5(t))) = \pi_4(\text{mg}_5(x_2^d x_5^t)) = x_3^{f_3} x_4^{f_4(t)} = x_3^{f_3} x_4^{h_4(t)+\delta_4}.$$

That is, we seek  $z_5(t)$  so that  $\pi_4(\mu_5(z_4(t), z_5(t))) = x_4^{\delta_4}$ . To that end, let us continue the path from  $z_4(t)$  on.

$$\begin{aligned} z_4(t) &= x_2^{d+f_3} x_5^{t-f_3} \xrightarrow{x_5^{t-f_3}} x_2^{d+f_3} x_4^{t-f_3} \\ &\xrightarrow{x_4^{\delta_4-1} x_5^{\binom{t-f_3}{2}-\binom{t-f_3-\delta_4+1}{2}}} x_2^{d+f_3} x_3^{\delta_4-1} x_4^{t-f_3-\delta_4+1}. \end{aligned}$$

The cost of the last step is again justified by Proposition 4.8. The 4-target is almost reached. One last step will allow to reach that target:

$$x_2^{d+f_3} x_3^{\delta_4-1} x_4^{t-f_3-\delta_4+1} \xrightarrow{x_4} x_2^{d+f_3} x_3^{\delta_4} x_4^{t-f_3-\delta_4}.$$

Since  $\pi_4(\mu_5(z_4(t), x_2^{d+f_3} x_3^{\delta_4} x_4^{t-f_3-\delta_4})) = x_4^{\delta_4}$ , we conclude

$$z_5(t) = x_2^{d+f_3} x_3^{\delta_4} x_4^{t-f_3-\delta_4}.$$

The cumulative cost from  $z_4(t)$  to  $z_5(t)$  is

$$\mu_5(z_4(t), z_5(t)) = x_4^{\delta_4} x_5^{(t-f_3)+\binom{t-f_3}{2}-\binom{t-f_3-\delta_4+1}{2}}.$$

Hence

$$\mu_5(x_2^d x_5^t, z_5(t)) = x_3^{f_3} x_4^{f_4(t)} x_5^{h_5(t)},$$

where

$$\begin{aligned} h_5(t) &= \binom{t+2}{3} - \binom{t+2-f_3}{3} + (t-f_3) + \binom{t-f_3}{2} - \binom{t-f_3-\delta_4+1}{2} \\ &= \binom{t+2}{3} - \binom{t+2-f_3}{3} + \binom{t+1-f_3}{2} - \binom{t-f_3-\delta_4+1}{2}. \end{aligned}$$

Again, we know that  $f_5(t) - h_5(t)$  is a constant, let us call it  $\delta_5$ . Thus

$$\delta_5 = f_5(t) - h_5(t) = f_5(0) - h_5(0).$$

We also know that  $\delta_5 = \tau_5(x_2^d) - \tau_4(x_2^d)$ . Now

$$\begin{aligned} f_5(0) &= \binom{d+2}{4}, \\ h_5(0) &= -\binom{2-f_3}{3} + \binom{1-f_3}{2} - \binom{-f_3-\delta_4+1}{2}. \end{aligned}$$

Using the formulas  $\binom{-a}{3} = -\binom{a+2}{3}$  and  $\binom{-b}{2} = \binom{b+1}{2}$ , we have

$$\begin{aligned} \delta_5 &= f_5(0) - h_5(0) \\ &= \binom{d+2}{4} + \binom{2-f_3}{3} - \binom{1-f_3}{2} + \binom{1-f_3-\delta_4}{2} \\ &= \binom{d+2}{4} - \binom{f_3}{3} - \binom{f_3}{2} + \binom{f_3+\delta_4}{2} \\ &= \binom{d+2}{4} - \binom{\binom{d}{2}}{3} - \binom{\binom{d}{2}}{2} + \binom{\binom{d}{2} + \binom{d+1}{3} + \binom{\binom{d}{2}}{2}}{2}. \end{aligned}$$

We are now in a position to get an explicit formula for  $\tau_5(x_2^d)$ . We have

$$\begin{aligned}\tau_5(x_2^d) &= \tau_4(x_2^d) + \delta_5 \\ &= \binom{d}{2} + \binom{d+1}{3} + \binom{d+2}{4} - \binom{\binom{d}{2}}{3} + \binom{\binom{d}{2} + \binom{d+1}{3} + \binom{\binom{d}{2}}{2}}{2} \\ &= \binom{\binom{\binom{d}{2}}{2} + \binom{d+1}{3} + \binom{d}{2}}{2} - \binom{\binom{d}{2}}{3} + \binom{d+3}{4} - d.\end{aligned}$$

□

**5.1. A conjecture.** Recalling the formulas of  $\tau_n(x_2^d)$  for  $n = 3, 4, 5$ , namely

$$\begin{aligned}\tau_3(x_2^d) &= \binom{d}{2}, \\ \tau_4(x_2^d) &= \binom{\binom{d}{2}}{2} + \binom{d+1}{3} + \binom{d}{2}, \\ \tau_5(x_2^d) &= \binom{\binom{\binom{d}{2}}{2} + \binom{d+1}{3} + \binom{d}{2}}{2} - \binom{\binom{d}{2}}{3} + \binom{d+3}{4} - d,\end{aligned}$$

we are led to the following conjecture.

**Conjecture 5.3.** *For all  $n \geq 3$ , the Gotzmann threshold  $\tau_n(x_2^d)$  is a polynomial of degree  $2^{n-2}$  in  $d$  with dominant term*

$$\frac{d^{2^{n-2}}}{2^{2^{n-2}-1}} = 2(d/2)^{2^{n-2}},$$

*i.e. the dominant term of the  $(n-2)$ -iterated binomial coefficient*

$$\binom{\binom{\binom{d}{2}}{\dots}{2}}{2}.$$

*Equivalently,*

$$\lim_{d \rightarrow \infty} \frac{\tau_n(x_2^d)}{\binom{\tau_{n-1}(x_2^d)}{2}} = 1.$$

By the above formulas, the conjecture holds true for  $n = 3, 4, 5$ .

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